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SPECTRAL DECOMPOSITION IN MODELS OF HEARING

by

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Abstract

In this paper we investigate some possibilities to construct an auditory filterbank based on models of hearing. Two specific methods are studied. The first one concerns a straightforward application of finite differences. This method can result in unacceptable numerical behaviour beyond the place where resonance takes place. A proposal is given to suppress this effect. The second method is an application of the Fredholm theory for the solution of differential equations. This method leads to a system of equations (filterbank) which is solved according to the theory of spectral decomposition.

1. INTRODUCTION

Frequency selectivity of the auditory system is one of the best-known characteristics of our organ of hearing. It is well known that this capacity originates from mechanical properties of the cochlea. Additional effects, such as combination tones (Ruggero et al., 1992a) and two-tone suppression (Ruggero et al., 1992b) seem to be the result of mechanical processes in which non-linearities play a leading part. As a consequence of this, it is necessary to model mechanical properties of the cochlea in the time domain. A second reason for time domain modelling follows from the recent interest in perceptive cues of dynamic speech signals.

For an overview of properties of models in relation to hearing theory, we refer to De Boer (1980, 1984, 1991). In the present paper we shall develop a dynamical system which has much in common with an auditory filterbank. Our point of departure is found in Van Dijk (1991). For reasons of convenience we first resume some of the ideas from that work.

We represent a small region of the basilar membrane at a distance x from the stapes by a harmonic oscillator (Fig. 1). The equation of motion of the oscillator is

$$m\ddot{u}(x,t)+r(x)\dot{u}(x,t)+k(x)u(x,t)=-2p(x,0,t) \quad , \qquad 0<x<l \quad . \tag{1}$$

m, $r(x)$ and $k(x)$ respectively are the effective mass, resistance and stiffness of the oscillator at x. $-2p(x,0,t)$ is the pressure difference across the membrane at that point. l is the length of the membrane and $u(x,t)$ is the deflection of the oscillator.

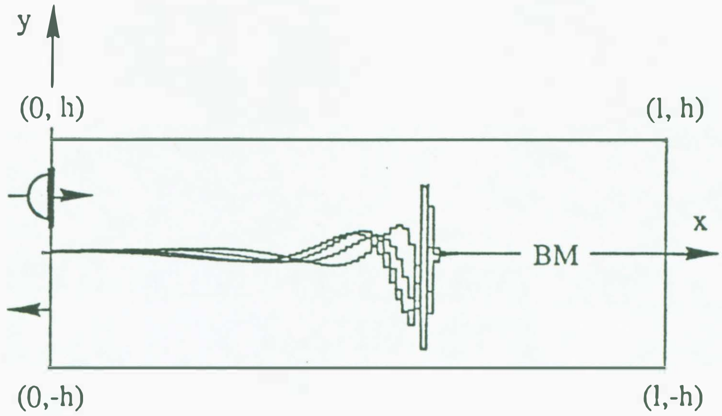

Figure 1. Model of the inner ear. The basilar membrane (BM) is the partition between the two cochlear scalae. Both scalae contain an ideal incompressible fluid. The propulsion of the system takes place at the stapes.

Differentiation with respect to the time is denoted by a dot. Consequently, $v(x,t) = \dot{u}(x,t)$ is the velocity of the oscillator at x. Differentiation with respect to x or y is denoted with a subscript. For example $\partial p / \partial x = p_x$. For computational purposes we convert (1) to the following equivalent first order system.

$$\begin{aligned} \dot{u}(x,t) &= v(x,t) \\ \dot{v}(x,t) &= -\omega_0^2(x)u(x,t) - \varepsilon\omega_0(x)v(x,t) - \frac{2}{m}p(x,0,t) \ , \quad 0 < x < l \ . \end{aligned} \tag{2}$$

$\omega_0^2(x) = k(x)/m$, $\varepsilon\omega_0(x) = r(x)/m$ and $\omega_0(x) = 2\pi f_0(x)$. $f_0(x)$ is the resonance frequency at x; ε is a small positive constant which controls the damping of an oscillator. m is a constant. We shall assume that $\omega_0(x)$ is strictly decreasing between 0 and l. Therefore, for $0 < x < l$, the system (2) is a set of tuned filters.

Assume that $u(x,t)$ and $v(x,t)$ are known at the time t. If the pressure $p(x,0,t)$ at that time is given for every x, the system can be integrated numerically over a small amount of time Δt. As a result of this, $u(x,t+\Delta t)$ and $v(x,t+\Delta t)$ are found for every x. If the pressure at that new time is known for every x, the system can be integrated again; etc. Consequently, the system actually works as a real time filterbank. In consequence of this way of reasoning it is 'only' necessary to find an explicit expression for the pressure. This pressure has to fulfil the essential physical principles of the inner ear.

2. THE PROBLEM OF THE PRESSURE

Membrane oscillators have been coupled to each other by deformable materials and the surrounding fluid. The influence of this coupling can be expressed by applying an equation of Euler at the membrane

$$\rho\ddot{u}(x,t) = -\frac{\partial p(x,0,t)}{\partial y} \ . \tag{3}$$

This equation simply states that the membrane always follows the motion of its deformable surroundings. Using (3) and the second equation in (2), we find that

$$\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{2}{m} p = \omega_0^2(x)u(x,t) + \varepsilon\omega_0(x)v(x,t) \ , \tag{4}$$

$$p = p(x,0,t) \ .$$

In view of our application, we slightly modify this equation. If we assume that

- the height of the cochlear scalae is small with respect to the length l, i.e. $h / l << 1$;
- at the bony walls of the scalae, a hard-wall boundary condition holds true;
- the fluid in the scalae is an ideal incompressible one, so that the pressure obeys Laplace's equation,

it can be shown (Van Dijk, 1990) that

$$p_y \approx h p_{xx} \ . \tag{5}$$

Let us insert (5) in (4) and multiply both members of the equation by $2 / m$. Then it readily appears that the pressure obeys the equation

$$pm_{xx} - a^2 pm = a^2 g(x,t) \ , \qquad 0 < x < l \tag{6}$$

in which $pm = pm(x,t)$ and

$$pm = \frac{2}{m} p(x,0,t) \ ; \quad a^2 = \frac{2\rho}{mh} \qquad \text{and}$$

$$g(x,t) = \omega_0^2(x)u(x,t) + \varepsilon\omega_0(x)v(x,t) \ . \tag{7}$$

We subject the pressure to the following boundary conditions

$$pm = \frac{2}{m} f(t) \qquad \text{at} \quad x = 0 \quad \text{and}$$

$$pm = 0 \qquad \text{at} \quad x = l \ . \tag{8}$$

The first condition models the propulsion at the stapes; the second one defines a vanishing pressure at $x=l$. The solution of (6) subjected to the boundary conditions (8) essentially solves our problem. In the next sections we will give an outline and discuss some properties of two methods, which can be used to find the solution of (6) and (8) explicitly.

3. APPLICATION OF FINITE DIFFERENCES

Let us divide the membrane in n oscillators of equal length $\Delta = l / n$. The midpoints of successive oscillators are the points x_i ; $i = 1,2,.....,n$ (Fig. 2). The first oscillator starts at x_0 and the last one ends at the point x_{n+1}.

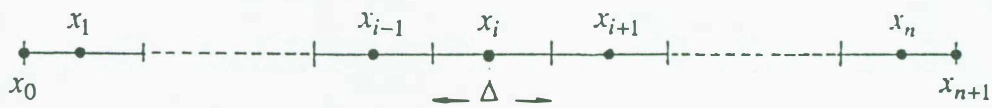

Figure 2. The discretization of the interval $0<x<l$ leads to oscillators with an effective width l/n.

At the point $x = x_i$; $i = 1,2,.....,n$ the system (2) reads

$$\begin{aligned} \dot{u}_i &= v_i \\ \dot{v}_i &= -\omega_{0i}^2 u_i - \varepsilon\omega_{0i} v_i - pm_i \end{aligned} \tag{9}$$

in which $u_i = u(x_i,t)$, $v_i = v(x_i,t)$, $\omega_{0i} = \omega_0(x_i)$, and $pm_i = 2/m\, p(x_i,0,t)$.
We shall assume that zero initial conditions complete the system. Thus we have

$$u_i = 0 \quad \text{and} \quad v_i = 0 \quad i = 1,2,.....,n \quad \text{at} \quad t = 0. \tag{10}$$

Let us assume for a while that all pm_i ; $i = 1,2, \quad ,n$ are prescribed functions of the time. Then the motion of the system (9) consists of the motion of n separate oscillators, which move under the influence of an known external pressure. This kind of problems can be integrated by application of standard numerical methods. In our work we applied a one-step integration method based on the fourth-order Runge-Kutta method. For details we refer to literature (for instance Press et al., 1986).

Except at the first and the last oscillator, the second order derivative of pm at all other oscillators is approximated by

$$pm_{xx} \approx \frac{pm_{i-1} - 2pm_i + pm_{i+1}}{\Delta^2} \qquad \text{at} \quad x_i \,; \quad i = 2,.....,n\text{-}1\,.$$

This approximation directly follows from the well-known forward and backward Taylor expansion. The truncation error is of the order $O(\Delta^4)$. Because the distance between x_0 and x_1 equals half the step size Δ, the approximation of pm_{xx} at x_1 must be slightly modified. Again from a forward and backward Taylor expansion, it follows in a straightforward way that

$$pm_{xx} \approx \frac{4}{3}\frac{2pm_0 - 3pm_1 + pm_2}{\Delta^2} \qquad \text{at} \quad x_1\,,$$

in which the truncation error is of the order $O(\Delta^3)$. For pm_{xx} at x_n we arrive at the similar expression

$$pm_{xx} \approx \frac{4}{3}\frac{pm_{n-1} - 3pm_n + 2pm_{n+1}}{\Delta^2} \qquad \text{at} \quad x_n\,,$$

in which the truncation error is again of the order $O(\Delta^3)$.

From (8) follows that the boundary conditions for pm are

$$pm_0 = \frac{2}{m} f(t) \quad \text{at} \quad x_0 \quad ,$$

$$pm_{n+1} = 0 \quad \text{at} \quad x_{n+1} \, .$$

Application of the finite difference approximation to (6) and the boundary conditions (8) yields the system of equations (11).

$$-\left(3 + \frac{3}{4} a^2 \Delta^2\right) pm_1 + pm_2 = \frac{3}{4} a^2 \Delta^2 g_1 - \frac{2}{m} f(t)$$

$$pm_{i-1} - \left(2 + a^2 \Delta^2\right) pm_i + pm_{i+1} = a^2 \Delta^2 g_i \; ; \; i = 2, \ldots\ldots, n-1 \qquad (11)$$

$$pm_{n-1} - \left(3 + \frac{3}{4} a^2 \Delta^2\right) pm_n = \frac{3}{4} a^2 \Delta^2 g_n$$

According to (7) g depends on x. At x_i we denote this quantity by

$$g_i = \omega_{0i}^2 u_i + \varepsilon \omega_{0i} v_i \quad i = 1, 2, \ldots\ldots, n \quad . \qquad (12)$$

The formal shape of (11) is

$$\mathbf{Ap} = \mathbf{b} \quad , \qquad (13)$$

in which $\mathbf{A}$ is a tridiagonal $n \times n$ matrix. The colomn vector $\mathbf{p} = \mathbf{p}(t)$ consists of the n components $pm_i = pm_i(t)$. Essentially, the components of the colomn vector $\mathbf{b} = \mathbf{b}(t)$ are the numbers g_i at the time t. The first component comprises an additional term, namely the input signal $f(t)$ at that time. (13) can be solved by application of the well-known tridiagonal algorithm (see for instance Press et al., 1986). The solution is the pressure vector $\mathbf{p}$ at the time t, so that every oscillator of the system (9) can be integrated over one time step Δt. An advantage of the present scheme is, that it is easy to realize by application of standard numerical procedures. However, numerical experiments show that the validity of results strongly depends on the behavior of the parameter function $\omega_0^2(x)$. Because $\omega_0^2(x) = k(x)/m$ and m is a constant, the scheme in essence depends on the behaviour of the stiffness along the membrane. In section 4 and 5 we shall discuss this point more closely. At this stage we first show an example of results from numerical experiments which are typical for the present scheme.

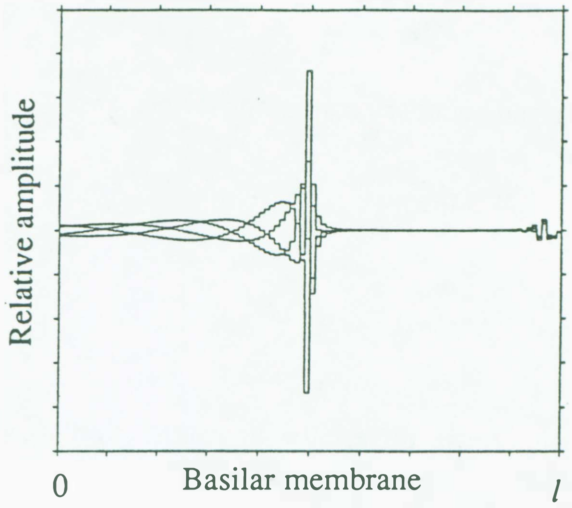

Figure 3. Four successive stages of a travelling wave phenomenon along the membrane. In the model the stiffness is an exponential decreasing function. The results show a numerical error at the endpoint of the system. In section 5.2 this error is ascribed to the exponential behaviour of the stiffness along the system.

4. REDUCTION OF THE MODEL

Let us reconsider the mathematical model for the pressure which follows from equation (6) and the boundary conditions (8). The model reads

$$pm_{xx} - a^2 pm = a^2 g(x,t)\ ,\ 0 < x < l$$

$$pm = \frac{2}{m} f(t) \qquad \text{at } x = 0 \tag{14}$$

$$pm = 0 \qquad \text{at } x = l\ .$$

$pm = pm(x,t)$ and $g(x,t)$ is given by (7). Zero initial conditions complete the model. Particularly, we shall assume that $f(t) = 0$ at $t=0$. The solution of problem (14) can be written as

$$pm(x,t) = \frac{2}{m}\frac{\sinh a(l-x)}{\sinh al} f(t) + q(x,t). \tag{15}$$

The first term in (15) is the contribution to the pressure due to the propulsion at $x = 0$. This term obeys the boundary conditions at $x = 0$ and $x = l$. The remaining part of (15), the function $q(x,t)$, has to fulfil the problem

$$q_{xx} - a^2 q = a^2 g(x,t)\,, \quad 0 < x < l$$

$$q = 0 \qquad \text{at } x = 0 \tag{16}$$

$$q = 0 \qquad \text{at } x = l\,.$$

The solution of this problem, contributes to the pressure *pm* only if membrane oscillators are in motion. This can be shown easily. Assume that the membrane oscillators are at rest. Then, according to (7), $g(x,t) = 0$. Because we assumed that $f(0) = 0$, the boundary conditions are zero too. In that case the solution of (16) is the zero solution. In consequence of this, $q(x,t)$ contributes to the pressure (15) only if the membrane oscillators are in motion. Therefore, this function describes all effects due to the coupling between the oscillators and is the crux of the whole matter. In the next section we first investigate problem (16) more closely. This will be done with the help of two specific examples.

5. TWO EXAMPLES

In (14) the function $g(x,t)$ is defined according to (7). This function not only depends on the motion of the oscillators, but also on the resonance frequency as a function of x. In consequence of this, we may expect that different functions $\omega_0(x)$ will lead to different characteristics in the coupling between the oscillators.
In our applications we are dealing with small values of the damping ε. This implies that the leading parameter function in $g(x,t)$ is $\omega_0^2(x)$. Therefore, we shall investigate the influence of $\omega_0^2(x)$ on problem (16) in the light of the following two examples.

5.1 The linear case

Let us start with a simple example. Assume that $\omega_0^2(x)$ is a linear function along the system which runs from the normalized value 1 at the point of propulsion x=0 to zero at x=l. All effects due to this kind of normalization can be met by time scaling and adjustment of the pressure with a constant multiplicative factor. In this case the squared resonance frequency is $\omega_0^2(x) = 1 - x/l$. Let us denote $r = 1 - x/l$ and let us consider r as the new independent variable of the problem. In terms of r, the squared resonance frequency reads $\omega_0^2(r) = r,\ 0 < r < 1$, and problem (16) takes the shape

$$q_{rr} - c^2 q = c^2 g(r,t)\,, \quad 0 < x < 1$$

$$q = 0 \qquad \text{at } r = 0 \tag{17}$$

$$q = 0 \qquad \text{at } r = 1\,,$$

in which

$$c = \frac{a}{l}\,.$$

From (7) follows that

$$g(r,t) = ru(r,t) + \varepsilon\sqrt{r}v(r,t), \tag{18}$$

5.2 The exponential case

Next we assume that $\omega_0^2(x) = \exp(-bx)$, $(b > 0)$, $0 < x < l$. In practice this case is often used because $\omega_0(x)$ defines a log-frequency to place map. Again we put $\omega_0^2(x) = r$ and conceive r as the new independent variable of the problem. Then problem (16) takes the shape

$$r^2 q_{rr} + rq_r - c^2 q = c^2 g(r,t) \ , \qquad \exp(-bl) < r < 1$$

$$q = 0 \qquad \text{at } r = \exp(-bl) \tag{19}$$

$$q = 0 \qquad \text{at } r = 1 \ .$$

The function $g(r,t)$ is the same as in the linear case and is given by (18). This time the constant c reads

$$c = \frac{a}{b} \ .$$

A typical value for the length of the system is $l = 3.5$ (cm). From literature (for instance De Boer, 1980) we know that $b \approx 3$. Thus, the order of the magnitude of b and l is the same. In consequence of this, the constant c in both problems is about the same. The order of $bl = 10$. However, $\exp(-bl) \approx 0.000045$, which means that $r = \exp(-bl)$ is effectively zero. The difference between problem (17) and problem (19) is that in (19) the point $r = 0$ is a singular point of the differential operator $r\partial / \partial r(r\partial / \partial r)q - c^2 q$. This means that in applications of problem (16) in which $\omega_0^2(x) = \exp(-bl)$ a direct numerical approach can lead to unreliable results near $x = l$.

5.3 A possible solution

There are several ways to solve this problem. One of them is to get around an explicit exponential description of the stiffness function. This can be accomplished if we restrict ourselves to a sectionally linear approximation of the stiffness along the membrane. An example of that approach is given in figure 4a. The function which has been plotted in this figure is

$$m\omega_0^2(x) = \sum_{n=1}^{N} y_n(x) U\big(y_n(x)\big) + const \ . \tag{20}$$

The straight lines $y_n(x)$; $n = 1,.....N$ are defined according to

$$y_n(x) = -m\frac{x}{l} + 1 \quad \text{, with slope } m = 2^{n-1}.$$

$U(y)$ is the well-known unit step function. N is an arbitrary integer.

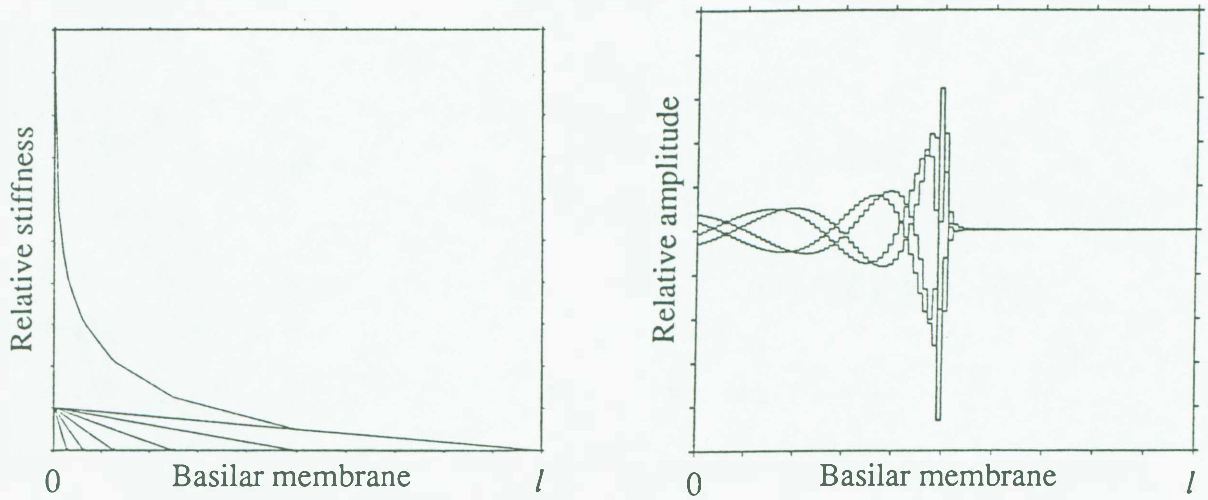

Figure 4. a. A sectionally linear approximation of an exponential stiffness function. The lower part of this figure shows the linear constituents of the approximation.
b. Results from the model, in which the stiffness is modelled according to figure 4a.

At the point of transition of two straight sections, the first derivative is discontinuous. If the magnitude of this discontinuity is not too large, the approximation (20) leads to satisfactory results. Fig. 4b is an example of results from (9) in which we introduced (20) as a model for the squared resonance frequency. The pressure pm_i; $i = 1,2,.....,n$, was determined by application of the scheme (11).

The main quality of the present method is that it works. The main disadvantage is that we lost the view of physical properties at the cost of calculational knacks. In section 6 we shall solve the same problem from a different point of view. In section 7 and 8 it will appear that this method is much closer related to the function of the cochlea as a frequency analysing system.

6. THE FREDHOLM ALTERNATIVE

In section 5 we noted that, mainly due to the small value of the damping constant ε, the stiffness of an oscillator is a leading parameter in our problem. Therefore, we shall reconsider our problem in the lossless case (i. e. $\varepsilon = 0$). From (2) follows that in this case the equation of motion for a single oscillator reads

$$\ddot{u}(x,t) = -\omega_0^2(x)u(x,t) - pm(x,t) \tag{21}$$

$0 < x < l$, $t \geq 0$. Again $pm(x,t)$ is defined in (7). According to section 4, $pm(x,t)$ can be written in the shape (15).

Insertion of (15) in (21) yields

$$\ddot{u}(x,t) = -\omega_0^2(x)u(x,t) - q(x,t) - \frac{2}{m}\frac{\sinh a(l-x)}{\sinh al}f(t) \ . \tag{22}$$

The unknown term $q(x,t)$ is the solution of problem (16) in the lossless case. By application of the Green's function technique (see for instance Morse and Ingard, 1986) this solution can be expressed explicitly as

$$q(x,t) = -a^2\int_0^l G(x,\xi,a)g(\xi,t)d\xi \ , \tag{23}$$

in which $G(x,\xi,a)$ is the solution of the problem

$$\begin{aligned} &G_{xx} - a^2G = -\delta(x-\xi) \ , \quad 0 < x < l \\ &G = 0 \qquad\qquad \text{at}\, x = 0 \ \text{ and } \ x = l \ . \end{aligned} \tag{24}$$

The function of Green which satisfies problem (23) is

$$G(x,\xi,a) = \begin{cases} \dfrac{\sinh ax \sinh a(l-\xi)}{a\sinh al} & 0 \le x < \xi < l \\ \\ \dfrac{\sinh a\xi \sinh a(l-x)}{a\sinh al} & 0 < \xi < x \le l \end{cases} \ . \tag{25}$$

According to (7) the function $g(x,t)$ simplifies in the lossless case to

$$g(x,t) = \omega_0^2(x)u(x,t) \ . \tag{26}$$

The Green's function $G(x,\xi,a)$ expresses the spatial extent of the coupling between membrane oscillators as an influence function, which results from a unit pressure with density 1 placed at the point $x = \xi$. Figure 5 is a plot of $G(l/2,\xi,a)$, $0 < \xi < l$. In this plot the constant $a = 20$. This value results from (7), where we used the typical model parameters $m = 0.05\ \text{g}/\text{cm}^2$; $\rho = 1\ \text{g}/\text{cm}^3$ and $h = 0.1$ cm.

(22), (23) and (26) constitute the integral equation

$$\ddot{u}(x,t) = -\omega_0^2(x)u(x,t) + a^2\int_0^l G(x,\xi,a)\omega_0^2(\xi)u(\xi,t)d\xi - \frac{2}{m}\frac{\sinh a(l-x)}{\sinh hal}f(t) \ , \tag{27}$$

$0 \le x \le l$; $t \ge 0$. This equation is a Fredholm integral equation of the second kind. In the sections 8 and 9 we shall solve (27) numerically. Before this is done, we first investigate some properties of (27).

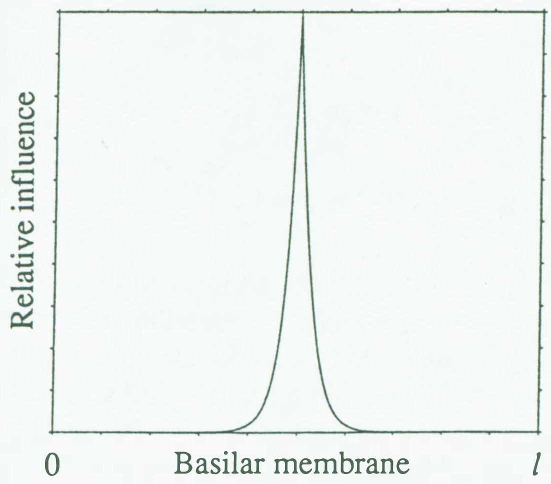

Figure 5 The coupling between oscillators can be expressed as an influence function. In this figure the influence function for the point $x=l/2$ is given. This function is the Green's function $G(l/2,\xi,a)$ in which $a=20$.

7. THE EQUIVALENT BOUNDARY VALUE PROBLEM

It often happens that Fredholm integral equations originate from equivalent boundary value problems. In our case, the integral in (27) follows from the boundary value problem (16). The formal shape of the solution of this problem is (23). This solution generates a time-invariant operator L defined by

$$L[\Phi(x)] = -\int_0^l G(x,\xi,a)\Phi(\xi)d\xi \ . \tag{28}$$

The inverse of L follows from problem (16) and reads

$$L^{-1}[\Phi(x)] = \Phi_{xx} - a^2\Phi, \quad 0 < x < l$$
$$\Phi = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = l \ . \tag{29}$$

Using the abridged notation which we defined in (28), (27) can be written as

$$\ddot{u}(x,t) + \omega_0^2(x)u(x,t) + \frac{2}{m}\frac{\sinh a(l-x)}{\sinh al} f(t) = -L\left[a^2\omega_0^2(x)u(x,t)\right] \tag{30}$$

Application of (29) to (30) yields the boundary value problem

$$\frac{\partial^2}{\partial x^2}\left(\ddot{u}(x,t) + \omega_0^2(x)u(x,t)\right) - a^2\ddot{u}(x,t) = 0 \ , \ 0 < x < l \ , \tag{31}$$

with boundary conditions

$$\ddot{u}(0,t) + \omega_0^2(0)u(0,t) = -\frac{2}{m}f(t) \quad \text{at } x = 0$$

$$\ddot{u}(l,t) + \omega_0^2(0)u(l,t) = 0 \quad \text{at } x = l \ .$$

The first boundary condition is the equation of motion for the deflection of the oscillator at $x = 0$. This oscillator moves under the influence of a known 'force' $-2/m f(t)$. The solution of this problem is quite elementary. Therefore, we represent this deflection simply by a known function $u_0(t)$. The second boundary condition is the equation of motion for the oscillator at $x = l$ in the absence of a driving 'force'. Because we assumed that all initial conditions are zero, the only solution of this equation is the vanishing one. In consequence of this, we put $u(l,t) = 0$. Thus the boundary conditions for (31) are

$$u = u_0(t) \quad \text{at} \quad x = 0$$

$$u = 0 \quad \text{at} \quad x = l \ . \tag{32}$$

Equation (31) is the differential equation for the membrane deflection. This equation can be considered as an modified wave equation. One important property of (31) readily follows if we assume that the time behaviour of the deflection is proportional to $\exp(-i\omega t)$. Therefore we introduce in (31)

$$u(x,t) = \bar{u}(x)\exp(-i\omega t) \ .$$

The resulting equation has the shape

$$\frac{\partial^2}{\partial x^2}\left(A(x,\omega)\bar{u}\right) + a^2\omega^2\bar{u} = 0 \ , \tag{33}$$

in which

$$A(x,\omega) = \omega_0^2(x) - \omega^2 \ . \tag{34}$$

If $\omega < \omega_0(x)$, $A(x,\omega)$ is positive and the solutions of (33) are oscillating. If $\omega > \omega_0(x)$, $A(x,\omega)$ is negative and the solutions of the equation are exponential functions. This means that $\omega = \omega_0(x)$ defines a turning point for equation (33).

Let us divide $A(x,\omega)$ by $-i\omega$ and multiply the result with m. Here m is the constant mass of an oscillator at x. Then we arrive at the definition of the impedance for the oscillator at x. In consequence of this, the product $A(x,\omega)\bar{u}$ equals the pressure at x normalized per unit of mass. Let us put

$$\bar{p} = A(x,\omega)\bar{u} \ , \tag{35}$$

and introduce (35) in (33). This yields the equation

$$\frac{\partial^2 \bar{p}}{\partial x^2} + \frac{a^2\omega^2}{A(x,\omega)}\bar{p} = 0 \qquad 0 < x < l . \tag{36}$$

We recognize in equation (36) a simplified version of Zwislocki's equation for the pressure in the cochlear scala. For the history of this equation -which can be derived under the so-called 'long-wave' assumption- we refer to Zwislocki (1980) or De Boer (1980). Some of its properties -under which also the meaning of a turning point in models of hearing- can be found in De Boer (1980).

8. AN EQUIVALENT DISCRETE SYSTEM

In this section we shall develop from the integral equation (26) a system of equations which can be solved numerically. In order to do that, we first consider the lossless case $(\varepsilon = 0)$. Note that at the points $x = x_i$; $i = 1,2,.....,n$ equation (21) constitutes a system of equations which can be written as

$$\ddot{\mathbf{U}} = -\mathbf{\Omega}^2\mathbf{U} - \mathbf{PM} \ . \tag{37}$$

$\mathbf{U} = \mathbf{U}(t)$ is a colomn vector with components $u_i = u(x_i,t)$; $i = 1,2,.....,n$.
Ω^2 is a $n \times n$ matrix which is diagonal. The elements are the n numbers $\omega_{0i}^2 = \omega_0^2(x_i)$; $i = 1,2,.....,n$. The vector **PM** is a colomn vector with components $pm_i = pm(x_i,t)$; $i = 1,2,.....,n$. As follows from (15), pm_i reads

$$pm_i = q(x_i,t) + \frac{2}{m}\frac{\sinh a(l - x_i)}{\sinh al} f(t) \ . \tag{38}$$

According to (23), $q(x_i,t)$ can be written as

$$q(x_i,t) = -a^2 \int_0^l G(x_i,\xi,a) g(\xi,t) d\xi \ . \tag{39}$$

We approximate the integral in (39) by the infinite sum

$$a^2 \int_0^l G(x_i,\xi,a) g(\xi,t) d\xi \approx a^2 \sum_{j=1}^{n} G(x_i,\xi_j,a) g(\xi_j,t) \Delta \ , \tag{40}$$

in which we assume that $\xi_i = x_i$; $i = 1,2,.....,n$. $\Delta = l / n$ is the width of a discrete oscillator and equals the step size of the discretization in figure 2. Let us insert (26) in (40). Then, it is readily seen that the approximation (40) constitutes a colomn vector **Q** with components $q(x_i,t)$; $i = 1,2,.....,n$ and can be written as

$$\mathbf{Q} = -\mathbf{G}\mathbf{\Omega}^2\mathbf{U} \ , \tag{41}$$

in which the elements of the matrix **G** are in which the elements of the matrix $\mathbf{G}$ are

$$G_{ij} = a^2 G\left(x_i, \xi_j, a\right)\Delta \;\; ; \;\; i,j = 1,2,.....,n \, .$$

From (25) follows that

$$G_{ij} = G_{ji} \;\; , \tag{42}$$

which implies that $\mathbf{G}$ is a (real) Hermitian matrix. Let us combine (38) and (41). The result is the colomn vector

$$\mathbf{PM} = -\mathbf{G}\Omega^2\mathbf{U} + \mathbf{P} \;\; , \tag{43}$$

in which the colomn vector $\mathbf{P} = \mathbf{P}(t)$ has the components

$$\frac{2}{m}\frac{\sinh a(l - x_i)}{\sinh al} f(t) \;\; ; \;\; i = 1,2,.....,n \;\; .$$

Insertion of (43) in (37) yields

$$\ddot{\mathbf{U}} = -(\mathbf{I} - \mathbf{G})\Omega^2\mathbf{U} - \mathbf{P} \;\; . \tag{44}$$

$\mathbf{I}$ is the identity matrix. System (44) is the numerical equivalent of the integral equation (27). We note that $\Omega^2 = \Omega \times \Omega$. Thus an equivalent shape of the system (44) is found if we premultiply each term of (44) by Ω. The result can be written as

$$\ddot{\mathbf{X}} = -\mathbf{AX} - \mathbf{F} \tag{45}$$

in which $\mathbf{X} = \mathbf{X}(t)$ and

$$\mathbf{X} = \Omega\mathbf{U} \;\; ,$$
$$\mathbf{A} = \Omega(\mathbf{I} - \mathbf{G})\Omega \;\; . \tag{46}$$

The colomn vector $\mathbf{F} = \mathbf{F}(t)$ is defined according to

$$\mathbf{F} = \Omega\mathbf{P} \;\; . \tag{47}$$

The system (45) describes the motion of n coupled oscillators, which move under the influence of an external vectorial ‘force’ $\mathbf{F} = \mathbf{F}(t)$. Because we assumed that in our problem initial conditions are zero, the problem is completed by putting

$$\mathbf{X}(0) = \mathbf{0} \qquad \dot{\mathbf{X}}(0) = \mathbf{0} \;\; . \tag{48}$$

Thus we have

$$\mathbf{A} = \mathbf{A}^t \quad , \tag{49}$$

in which $\mathbf{A}^t$ is the transposed of $\mathbf{A}$. (49) implies that $\mathbf{A}$ is a (real) Hermetian matrix.

9. SPECTRAL DECOMPOSITION

We first consider the homogeneous part of (45). This system reads

$$\ddot{\mathbf{X}} = -\mathbf{A}\mathbf{X} \quad .$$

Let us look for solutions in the shape

$$\mathbf{X}(t) = \Phi e^{i\sqrt{\lambda}t} \quad .$$

Substitution in the system yields the eigenvalue problem

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{0} \quad .$$

We solved this problem by application of a recommended 'Eispack' path (Smith et al., 1976). From the eigenvectors we formed a matrix $\mathbf{T}$. The j-th colomn of $\mathbf{T}$ is the normalized eigenvector which belongs to the j-th eigenvalue λ_j ; $j = 1,2,\ldots\ldots,n$. It is known from linear algebra (see for instance Berberian, 1992) that $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1} \quad , \tag{50}$$

in which the diagonal matrix $\mathbf{D}$ is defined by

$$\mathbf{D} = \begin{vmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{n-1} & \\ & & & & \lambda_n \end{vmatrix}$$

The diagonal elements are the eigenvalues λ_i ; $i = 1,2,\ldots\ldots,n$. Because $\mathbf{A}$ is a symmetrical matrix, $\mathbf{T}$ has the property that

$$\mathbf{T}\mathbf{T}^t = \mathbf{I} \quad ,$$

which means that in our problem all eigenvectors are orthonormal. In consequence of this we have

$$\mathbf{T}^t = \mathbf{T}^{-1} \ .$$

Therefore, an equivalent reading of (50) is

$$\mathbf{A} = \mathbf{TDT}^t \ . \tag{51}$$

Let us insert (51) in (45) and premultiply each term of the system by $\mathbf{T}^t$. Then the system takes the shape

$$\ddot{\mathbf{Y}} = -\mathbf{DY} - \mathbf{H} \ . \tag{52}$$

$\mathbf{Y} = \mathbf{Y}(t)$ and $\mathbf{H} = \mathbf{H}(t)$. The colomn vectors $\mathbf{Y}$ and $\mathbf{Z}$ are related to each other by the transforms

$$\mathbf{Y} = \mathbf{T}^t\mathbf{X} \quad \text{or} \quad \mathbf{X} = \mathbf{TY} \ . \tag{53}$$

In the same manner holds

$$\mathbf{H} = \mathbf{T}^t\mathbf{F} \quad \text{or} \quad \mathbf{F} = \mathbf{TH} \ . \tag{54}$$

According to (48) and (53), the initial conditions for the system (52) are

$$\mathbf{Y}(0) = \mathbf{0} \ \text{ and } \ \dot{\mathbf{Y}}(0) = \mathbf{0} \ . \tag{55}$$

The system (52) describes the motion of n uncoupled oscillators, each of which is forced to move under the influence of an external known 'force'. The force for the i-th oscillator is the i-th component of the known colomn vector $\mathbf{H}$. The system performs its motion from a state of rest. The elements of the diagonal of $\mathbf{D}$ are the squared resonance frequencies of successive oscillators. In our application we arranged the eigenvalues in a decreasing order. Thus the system represents a filterbank. In order to implement the present results, we first write (52) as a first order system defined by

$$\dot{\mathbf{Y}} = \quad \mathbf{Z}$$

$$\dot{\mathbf{Z}} = -\mathbf{DY} - \mathbf{H} \ .$$

After that, we introduced a small amount of damping in every oscillator. The damping is proportional to the resonance frequency of an oscillator. In vector notation the damping term reads

$$-\varepsilon\sqrt{\mathbf{D}}\mathbf{Z} \ ,$$

in which ε is a small positive number. Thus, the ultimate system has the shape

$$\dot{\mathbf{Y}} = \mathbf{Z}$$
$$\dot{\mathbf{Z}} = -\mathbf{D}\mathbf{Y} - \varepsilon\sqrt{\mathbf{D}}\mathbf{Z} - \mathbf{H} \tag{56}$$

with initial conditions

$$\mathbf{Y}(0) = \mathbf{0} \text{ and } \mathbf{Z}(0) = \mathbf{0} \ .$$

In order to integrate (56) in the time domain, we again applied a fourth-order Runge-Kutta method. The deflection vector $\mathbf{Y}$, transformed according to (53) and (46) leads to the deflection of the original system (37). Figure 6 is one of the results of this method.

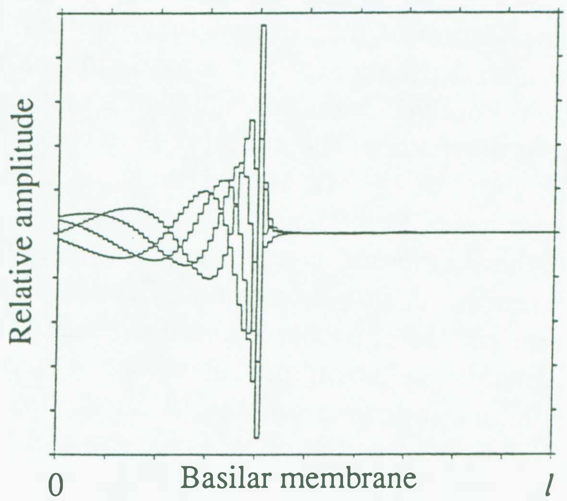

Figure 6. The mapping of the motion of a system of uncoupled filters, which move in an n-dimensional Euclidian space, to the 'real' world. In the model the stiffness is an exponential decreasing function.

10. DISCUSSION

We considered the basilar membrane as a series of tuned oscillators. In the absence of any kind of coupling between successive oscillators, the system is an elementary filterbank. A forced motion of such a system can be found by applying rather standard numerical procedures. In our application, we systematically applied a fourth-order Runge-Kutta one-step method.

When we reckon the fluid-like environment of the basilar membrane, an indirect coupling between all oscillators leads to a description of a non-homogeneous transmission line. In that case the equation of motion for the deflection of the oscillators is eq. (31). This equation is a modified wave equation. After transformation of (31) to the frequency domain, the existence of a turning point is readily established. A simple substitution shows that the equation for the deflection in the frequency domain is qualitatively comparable with Zwislocki's equation for the pressure in the cochlear fluid (Zwislocki, 1980). This clearly states the relation with classical hearing theory.

The problem for the pressure in the time-domain has been posed in section 2. This problem is the time-domain counterpart of Zwislocki's equation.

In section 3 we applied, completely uncritically, finite differences to the problem of the pressure and solved this problem. The motion of the membrane which results from this pressure shows that the naive approach can lead to unreliable numerical results after the place where resonance takes place. The origin of this unwelcome behaviour is the path of the stiffness function (in our model the squared resonance frequency) along the membrane. In section 5.2 we showed that if the stiffnes is an exponential (decreasing) function, the differential operator which governs the coupling between the oscillators is almost singular. A possibility to avoid this problem is given in section 5.3. There we propose to replace an exponential stiffness function by a sectionally linear approximation. An example of that approximation is given and leads to satisfactory numerical results.

There is more in life than finite differencing. Therefore, we studied in the next sections the same problem from a different point of view. We first solved the problem for the pressure by application of the Green's function technique. This technique leads to a Fredholm integral equation of the second kind for the deflection of the oscillators. After that, we derived from this equation a system of equations for the deflection of n discrete oscillators. The coupling is expressed by a time invariant Hermitian matrix. In that case, it is natural to apply the concept of spectral decomposition (see for instance Berberian, 1992) to the system. The result shows that the actual motion of the membrane is an orthogonal mapping of a qualitatively equivalent dynamical process in an n-dimensional Euclidian space. In this space we have only to consider the motion of uncoupled oscillators. Each oscillator performs its motion under the influence of an known force along one of the axes in that space. The tuning of those oscillators follows from the solution of a classical eigenvalue problem. The normalized eigenvectors of this problem constitute the orthonormal transformation matrix. A technical restriction to this solution, follows from the condition number of the matrix in the eigenvalue problem and limits the frequency range of a filterbank.

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