# PARADOXES AROUND THE EXPONENTIAL HORN

## Are they danger signals ?

by

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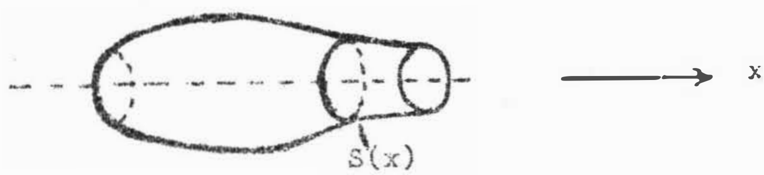
for Gerold Ungeheuer

For the calculation of slender acoustic tubes the equation of Webster[*)] is widely accepted. Adopting the velocity potential $\phi$ as a means for deriving sound pressure p and particle velocity u in the following way

$$p(x,t) = -\rho \frac{\partial \phi}{\partial t} \qquad u(x,t) = \frac{\partial \phi}{\partial x} \qquad (1)$$

where $\rho$ is the density of the medium ,
we have only to solve

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{S}\frac{dS}{dx}\frac{\partial \phi}{\partial x} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (2)$$




F I G U R E 1

The tube with non-uniform cross-area

in order to kill two birds with a stone.

For the exponential horn we shall study now, the cross-area S(x) depends on x as follows

$$S(x) = S_o e^{mx} \qquad (3)$$

Substitution of (3) in (2) yields

$$\frac{\partial^2 \phi}{\partial x^2} + m \frac{\partial \phi}{\partial x} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (4)$$

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*) A.G. Webster, Proc. Natl. Acad. Sci. (U.S.) 5, 275 - 282 ( 1919 )

The general solution of this equation can be found by means of the method of separating the variables $x$ and $t$

$$\phi(x,t) = \varphi(x)\psi(t) \qquad (5)$$

where $\varphi$ is a function only of $x$ and $\psi$ depends only on $t$. These two functions can be found by substitution of (5) in (4) and taking into consideration the boundary conditions: in other words by specifying how the tube is loaded at both ends and by what sound source it is being driven.

In engineering practice one is not always interested in the most general solution : it is customary to calculate and measure networks by seeing how they react to sinusoidal driving forces. It is a fact that, when a sinusoidal driving force is switched to a network, the voltages and currents in the meshes of that network will, in the long run, that is in the steady state after the transient phenomena have died down at last, have become sinusoidal too. What goes for electrical networks goes for acoustical networks too. So, if $\phi(x,t)$ of (5) is a general solution indeed, it must of necessity hold for the steady sinusoidal state. In other words, for this state

$$\phi(x,t) = \varphi(x) \sin \omega t \qquad (6)$$

or

$$\phi(x,t) = \varphi(x) \cos \omega t \qquad (7)$$

or even, in the complex notation

$$\phi(x,t) = \varphi(x)\, \varepsilon^{j\omega t} \qquad (8)$$

Substitution of (8) in (4) produces a differential equation in $\varphi(x)$ :

$$\frac{d^2\varphi}{dx^2} + m\frac{\partial\varphi}{dx} + \frac{\omega^2}{c^2}\varphi = 0 \qquad (9)$$

with the general solution

$$\varphi(x) = A_1\, \varepsilon^{b_1 x} + A_2\, \varepsilon^{b_2 x} \qquad (10)$$

where $A_1$ and $A_2$ take care of the boundary conditions and

$b_1$ and $b_2$ are the roots of :

$$b^2 + m\,b + \frac{\omega^2}{c^2} = 0 \qquad (11)$$

$$b_1 = -\frac{m}{2} + j\sqrt{\frac{\omega^2}{c^2} - \frac{m^2}{4}} \qquad (12)$$

$$b_2 = -\frac{m}{2} - j\sqrt{\frac{\omega^2}{c^2} - \frac{m^2}{4}} \qquad (13)$$

Let us, for the sake of simplicity, introduce the auxiliary velocity

$$v = \frac{c}{\sqrt{1 - \frac{m^2 c^2}{4\omega^2}}} \qquad (14)$$

then combination of (8), (10), (12), (13) and (14) will finally result in the complete solution

$$\phi(x,t) = A_1 \epsilon^{-\frac{1}{2}mx}\,\epsilon^{j\omega\left(t + \frac{x}{v}\right)} + A_2 \epsilon^{-\frac{1}{2}mx}\,\epsilon^{j\omega\left(t - \frac{x}{v}\right)} \qquad (15)\ *)$$

← v (wave 1) v → (wave 2) → x

Interesting enough , this equation presents the steady sinusoidal state as the superposition of two travelling waves running in opposite directions:

wave 1, travelling at the speed $v$ in the negative direction of $x$

wave 2, travelling at the speed $v$ in the positive direction of $x$.

Strange as it may seem for frequencies above $\omega_0 = \frac{1}{2}\,m\,c$ , the speed $v$ is supersonic :

$$\omega > \tfrac{1}{2}\,m\,c \quad , \quad v > c \; .$$

For $\omega = \frac{1}{2}\,m\,c$ , $v = \infty$ .

Finally , for $\omega < \frac{1}{2}\,m\,c$ there can be no wave propagation at all, suggesting a low-frequency cut-off at $\omega_0 = \frac{1}{2}\,m\,c$ .

Seemingly , we are saddled with two paradoxes:

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*) We would, of course, have obtained exactly the same result if we had announced , from the beginning,

$\phi(x,t) = A\,\epsilon^{bx}\,\epsilon^{j\omega t}$ as the solution of (4) for sinusoidal vibrations.

Paradox I ( the supersonic paradox ) : the solution for $\phi(x,t)$ 'contains' two waves travelling in opposite directions at a speed $v > c$ .

Paradox II ( the breath-taking paradox ) : the travelling waves are subject to a cut-off frequency that prevents the horn from transmitting the frequency $\omega = 0$ , that is a constant air flow like the breathstream.

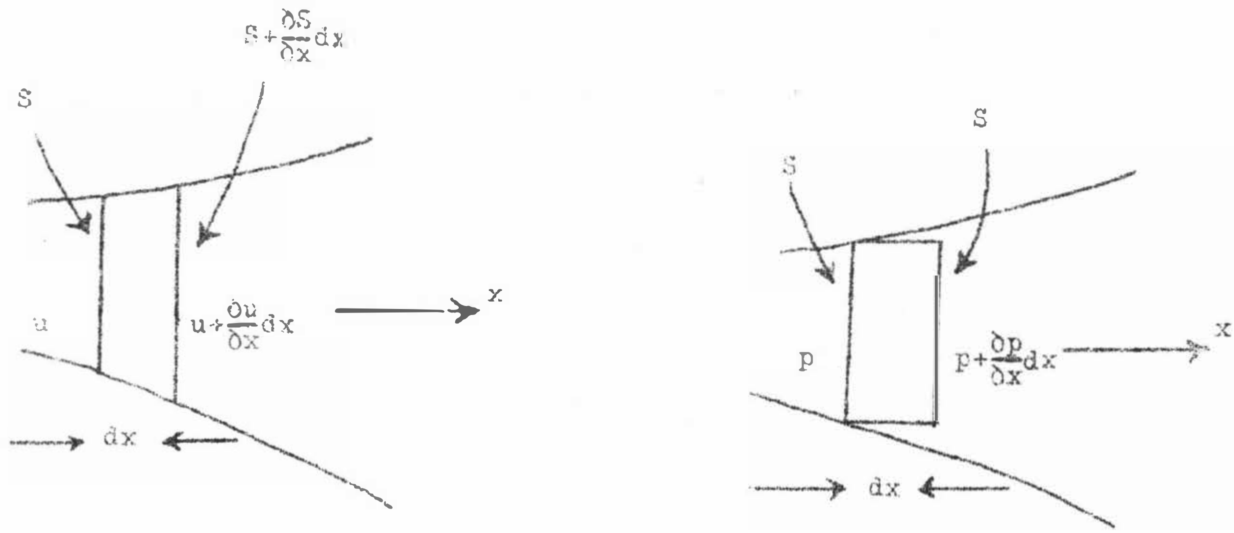
Whether we are alarmed by these paradoxes or not depends on the view we adopt; when we take the practical view we need not be alarmed as we may argue that the supersonic waves are nothing but shady characters in a mathematical shadow-show: they do not 'exist' physically. In this respect it is rewarding to consider the special case of $m = 0$ .

For $m = 0$ the exponential horn degenerates into a tube with constant cross-area, the well-known simple organ pipe. In this case the solution folds down to

$$\phi(x,t,) = A_1 e^{j\omega\left(t+\frac{x}{c}\right)} + A_2 e^{j\omega\left(t-\frac{x}{c}\right)} \qquad (16),$$

again the superposition of two travelling waves, but now travelling at the normal velocity of sound $c$ . This normal velocity makes these waves look less conspicuous and therefore less suspicious but, nevertheless, they do not physically exist either. There is no reason , however , to discard the elegant mathematical possibility of decomposing a stationary wave-pattern into two fictitious waves travelling in opposite directions. As will be shown in a special appendix ( see page 9 ) , the paradoxes do not prevent us from deriving the general circuit parameters of the exponential horn seen as an acoustical four terminal network. In addition, our sinusoidal steady state method leads to exactly the same formants, in the case of the exponential horn, as Ungeheuer's eigenvalue method. Practical considerations , however , should not sing us to sleep. Therefore, we may also adopt the view of fundamental criticism and ask ourselves : why does application of Webster's equation in the exponential case lead to the supersonic paradox ; is this a chance hit or is there something fundamentally wrong with Webster's equation that comes to the fore in a dramatic way in the case of the exponential horn ?

In order to pave the way to a possible answer to this question we shall first of all derive Webster's horn equation in the classical way, see fig. 2 .




application of the law of continuity

$$\rho \frac{\partial (Su)}{\partial\ x} = - S \frac{\partial \rho}{\partial t}$$

application of the dynamic law

$$\frac{\partial p}{\partial x} = - \rho \frac{du}{dt}$$

application of the adiabatic law

$$p = c^2 s$$

F I G U R E 2

Derivation of Webster's horn equation.

The acoustic calculations are based on three laws. One of these , the law of continuity , demands that in the horn no matter can be created nor annihilated. It is customary to apply this principle to the thin disc depicted in the left-hand figure. Per unit of time the following mass enters the surface S of the disc at right angles ( because of the one-dimensional strait jacket into which the problem is squeezed in the Webster method ) :

$$\rho \; u \; S \tag{17}$$

Through the surface $S + \frac{\partial S}{\partial x} dx$ there escapes , again per unit of time, the mass

$$\rho \; (u + \frac{\partial u}{\partial x} dx)(S + \frac{\partial S}{\partial x} dx) \tag{18}$$

By subtracting (17 ) from (18), at the same time neglecting terms containing $(dx)^2$ , we see that, seemingly, the disc produces , per unit of time , the mass

$$\rho \; \frac{\partial( \; Su \; )}{\partial x} \; dx \tag{19}$$

As creation of mass is not allowed the mass described by (19) has been obtained at the cost of the density in the disc; per unit of time the disc loses the mass

$$- S \; \frac{\partial \rho}{\partial t} \; dx \tag{20}$$

As (19) must be equal to (20) we arrive at

$$\boxed{\rho \; \frac{\partial (Su)}{\partial x} \; = \; - S \; \frac{\partial \rho}{\partial t}} \tag{21}$$

This equation contains the crux of Webster's method : the cross-area S has been elegantly included in the partial differential quotient. But how about the cost of elegance ? Webster's method implies that the velocity u is uniform over the cross-area of the tube and , moreover , that the streamlines enter and leave the disc at right angles without offering an explanation of how such a miracle might be accomplished by goings-on in the disc. Webster has tampered with the streamlines, transforming the situation into a caricature.

The next traditional step is to apply the dynamic law to the (slightly different ) disc in the right-hand figure. This law pertains to virtual displacements of the air particles, that is displacements along the streamlines. The thin cylinder is thought to move under the influence of a force furnished by the pressure difference exerted on the end surfaces. The surplus force in the positive direction of x equals

$$- S \frac{\partial p}{\partial x} dx \qquad (22)$$

The product of mass and acceleration of the cylinder is equal to

$$\rho S \, dx \frac{du}{dt} \qquad (23)$$

The dynamic law requires that (22) is equal to (23) , which leads to

$$\boxed{\frac{\partial p}{\partial x} = -\rho \frac{du}{dt}} \qquad (24)$$

We direct the attention of the reader to the fact that in the right-hand member there appears a total derivative.
Expression (24) is not a typical Webster invention because S does not appear in it. Nevertheless , sound pressure p is supposed to be uniform over the cross-area S whereas the air particles are supposed to move along streamlines that are parallel to the x - axis. In other words, also the dynamic law is applied to a caricature of the streamlines.

In order to be able to formulate the adiabatic law it is supposed that the density $\rho$ of the medium performs very small variations around it rest value $\rho_o$ in the following way

$$\rho = \rho_o + s \qquad (25)$$

The small quantity s is called the condensation. The adiabatic law says

$$\boxed{p = c^2 s} \qquad (26) .$$

Now the scene is set for the derivation of Webster's equation. We must take measures to ensure that we arrive at a linear differential equation. To begin with, we replace $\rho$ by $\rho_o$ and

$\frac{\partial \rho}{\partial t}$ by $\frac{\partial s}{\partial t}$ in (21) , so that we obtain

$$\rho_o \frac{\partial (Su)}{\partial x} = - S \frac{\partial s}{\partial t} \qquad (27)$$

The next step is to linearize (24) by replacing the total derivative of $u$ with respect to $t$ by the partial derivative. In general, we have

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx \qquad (28)$$

or

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u \qquad (29).$$

For the small velocities we meet in practice the second term of the right-hand member of (29) may be neglected. We then obtain

$$\frac{\partial p}{\partial x} = - \rho_o \frac{\partial u}{\partial t} \qquad (30),$$

at the same time replacing $\rho$ by $\rho_o$ .

We are now in a position to introduce the velocity potential $\phi$ defined in the following manner

$$p = - \rho_o \frac{\partial \phi}{\partial t} \qquad (31)$$

$$u = \frac{\partial \phi}{\partial x} \qquad (32).$$

This choice is based on the dynamic law (30) because when we substitute (31) and (32) in (30) we find that

$$- \rho_o \frac{\partial^2 \phi}{\partial t \partial x} = - \rho_o \frac{\partial^2 \phi}{\partial x \partial t} \quad \text{indeed.} \qquad (33)$$

By combining (26), (27), (31) and (32) we easily arrive at

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{S} \frac{\partial S}{\partial x} \frac{\partial \phi}{\partial x} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0} \qquad (34)$$

Here is Webster's horn equation at last.

It describes a loss-free , one-dimensional <u>model</u> of the general, slender acoustic tube. Its derivation is the result of an idealization of the streamlines , followed by the usual linearization for small amplitudes.

When is a model a good model, in spite of the fact that it does not fit physical reality in the tube to a T ? The answer depends on the criterion one chooses for judging the worthwhileness of a certain model. For phonetic applications with a view to the vocal tract one is inclined to say that the Webster model should produce the correct formants, taking for granted one knows how to <u>define</u> and how to <u>measure</u> the formants. In this respect no high accuracy may be expected nor needed as the speech code is based on contrasts between formants rather than on absolute positions of the formants. As far as I can judge at the moment the Webster model is a workable model but this does not remove my grudge against the supersonic paradox. I am toying with the idea , that it is the tampering with the streamlines that is at the root of the supersonic paradox. This idea is supported by the case of the conical horn we shall elaborate to some extent in the appendix ( see page 13 ). Interestingly enough , the hypothetical travelling waves in the conical horn travel at the normal speed c , in spite of the fact that we apply Webster's horn equation.

## A P P E N D I X

### 1. The general circuit parameters etc. of the exponential horn.

We shall first of all combine (1) , (8) and (10) which yields

$$p = -\rho \frac{\partial \phi}{\partial t} = -j\omega\rho \left[ A_1 \epsilon^{b_1 x} + A_2 \epsilon^{b_2 x} \right] \epsilon^{j\omega t} \qquad (35)$$

$$U = S \frac{\partial \phi}{\partial x} = S \left[ A_1 b_1 \epsilon^{b_1 x} + A_2 b_2 \epsilon^{b_2 x} \right] \epsilon^{j\omega t} \qquad (36) .$$

The reader will have noticed we have introduced the volume velocity

$$U = S\, u \qquad (37)$$

The constants $A_1$ and $A_2$ may be related to the values of p and U for x = 0 . These values depend on the way of driving as well as on the way of loading the horn. Before we do so, we take the liberty to omit the factor

$$\epsilon^{j\omega t} ,$$

keeping well in mind to re-introduce it whenever necessary . In that way p and U assume the character of complex amplitudes. For the sake of simplicity we shall not change their notation. So, for x = 0 we have

$$p_o = - j\omega\rho \left[A_1 + A_2\right] \tag{38}$$

$$U_o = S_o \left[A_1 b_1 + A_2 b_2\right] \tag{39}$$

Solution of (38) and (39) yields

$$A_1 = - \frac{b_2 \dfrac{p_o}{j\omega\rho} + \dfrac{U_o}{S_o}}{b_2 - b_1} \tag{40}$$

$$A_2 = \frac{b_1 \dfrac{p_o}{j\omega\rho} + \dfrac{U_o}{S_o}}{b_2 - b_1} \tag{41}$$

The next step is to substitute (40) and (41) in (35) and (36) , at the same time omitting the factor $\epsilon^{j\omega t}$ . As, in four pole theory, we are interested in the relation between the quantities at the sending end and those at the receiving end, we put $x = 1$ , the length of the horn. We then get, after some elaboration

$$p_1 = \frac{b_2 \epsilon^{b_1 1} - b_1 \epsilon^{b_2 1}}{b_2 - b_1} p_o + \frac{j\omega\rho}{S_o} \frac{\epsilon^{b_1 1} - \epsilon^{b_2 1}}{b_2 - b_1} U_o \tag{42}$$

$$U_1 = j\,\frac{S_1}{\omega\rho}\,b_1 b_2\,\frac{\epsilon^{b_1 l} - \epsilon^{b_2 l}}{b_2 - b_1}\,p_o + \frac{S_1}{S_o}\,\frac{b_2\epsilon^{b_2 l} - b_1\epsilon^{b_1 l}}{b_2 - b_1}\,U_o \qquad (43)$$

We may write these two expressions as follows , at the same time defining the general circuit parameters A , B , C and D :

$$p_1 = D\,p_o - B\,U_o \qquad (44)$$

$$U_1 = -\,C p_o + A\,U_o \qquad (45)\ .$$

It is easy to prove that

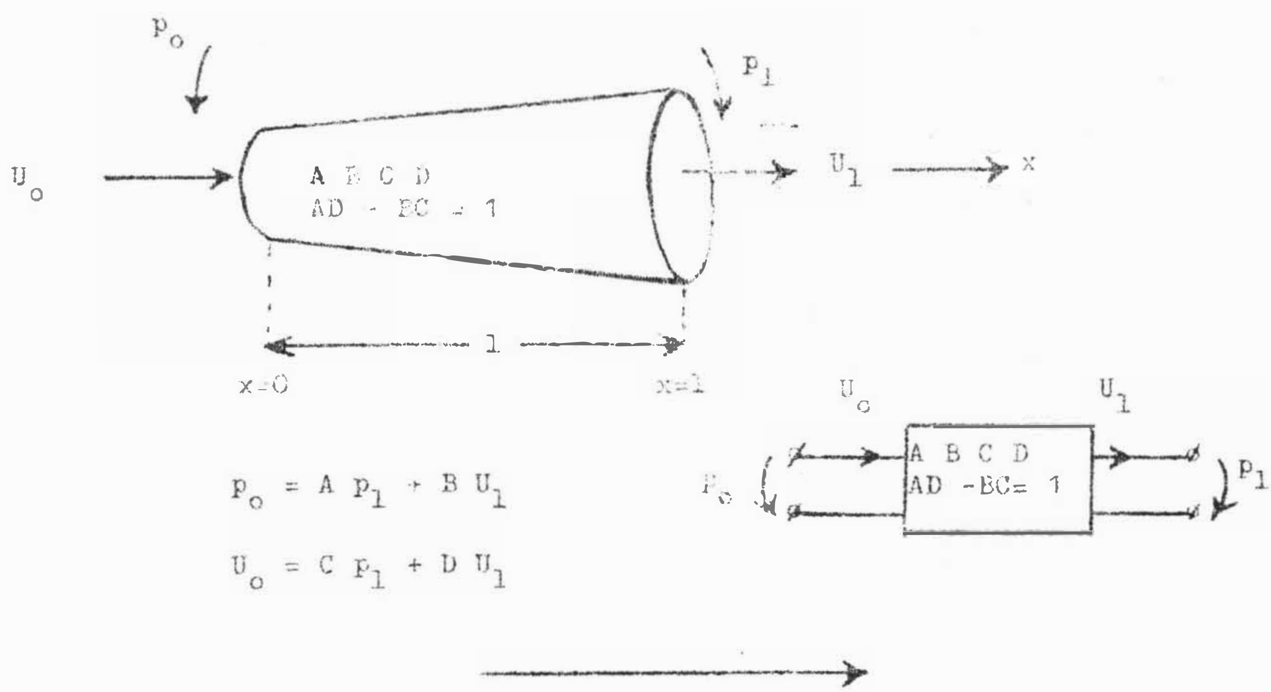
$$A\,D - B\,C = 1 \qquad (46)\ ,$$

which property allows us to solve $p_o$ and $U_o$ from (44) and (45) as follows

$$p_o = A\,p_1 + B\,U_1 \qquad (47)$$

$$U_o = C\,p_1 + D\,U_1 \qquad (48).$$

These are the well-known four-pole equations for sending from $x = 0$ to $x = l$ , see fig. 3 . They permit us, among other




direction of sending

F I G U R E 3

The (exponential ) horn seen as a four terminal network.

things, to predict the behaviour of ( exponential ) horns in cascade.

It is possible to solve the volume velocity $U_1$ in the mouth opening from (47) and (48) by introducing the radiation impedance $Z_1$ of the mouth opening , defined as

$$p_1 = U_1 Z_1 \tag{49}$$

and the internal impedance $Z_o$ of the throat defined as

$$p_o = e - U_c Z_o \tag{50}$$

where e represents the ' acousto-motoric force ' of the throat .
In phonetic practice, however, it is not necessary to go all the way. There one goes to the extremes of supposing $p_1 = 0$ in the mouth opening and $U_o = 0$ at the throat. Equation (48) shows that these assumptions make sense only for those frequencies for which $D = 0$ . As (45) clearly indicates these are the frequencies for which

$$b_2 e^{b_2 l} - b_1 e^{b_1 l} = 0 \tag{51}$$

By substituting (12) and (13) in (51) we finally arrive at

$$\tan \sqrt{\frac{\omega^2}{c^2} - \frac{m^2}{4}}\, l + \frac{2}{m}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2}{4}} = 0 \tag{52}$$

This is exactly the same formula as reached by means of the eigenvalue method .*)

It is interesting to notice , that the paradoxal cut-off frequency $\omega_o = \frac{1}{2} m c$ comes to the fore here as a real formant as it obeys expression (52) .

For $m = 0$ . the case of the tube with constant cross-area , (52) reduces to

$$\cos \frac{\omega l}{c} = 0 \text{ , as it should do.}$$

Moreover, (52) is sensitive to the sign of m , also a necessary property.

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*) Gerold Ungeheuer, Elemente einer akustischen Theorie der Vokalartikulation ( Springer-Verlag 1962 ).

## 2. The case of the conical horn.

In the case of the conical horn

$$S = a\,x^2 \tag{53}$$

so that Webster's equation reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{2}{x}\frac{\partial \phi}{\partial x} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{54}$$

Note that the constant $a$ does not appear in the wave equation.

For the steady sinusoidal state we have

$$\phi(x,t) = \varphi(x)\,\varepsilon^{j\omega t} \tag{55}$$

leading to

$$\frac{d^2\varphi}{dx^2} + \frac{2}{x}\frac{d\varphi}{dx} + \frac{\omega^2}{c^2}\varphi = 0 \tag{56}$$

with the solution

$$\varphi(x) = \frac{A_1}{x}\,\varepsilon^{j\frac{\omega}{c}x} + \frac{A_2}{x}\,\varepsilon^{-j\frac{\omega}{c}x} \tag{57}$$

so that

$$\phi(x,t) = \frac{A_1}{x}\,\varepsilon^{j\omega(t+\frac{x}{c})} + \frac{A_2}{x}\,\varepsilon^{j\omega(t-\frac{x}{c})} \tag{58}$$

c ⟵ ⟶ c

This equation again confronts us with two ( fictitious ) waves travelling in opposite directions but at the normal speed $c$ . We must be very careful in drawing our conclusions now ! As regards the tampering with the streamlines Webster's model is no better for the conical horn than for the exponential horn. It leads, however, to a wave equation that produces no supersonic paradox. This wave equation , by the way, inspires us with a better model for the conical horn: when we replace the linear variable $x$ in (54) by the radial variable $r$ we get :

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \qquad (59).$$

This is the well-known wave equation for spherical waves unmarred by the supersonic paradox. The conical horn deserves a model better than the Webster model: the spherical model with wave propagation in radial directions and spherical wave fronts.

So, in my opinion we have learned the following lesson: we should adapt the model to the shape of the horn and not, like Webster did, squeeze an arbitrary shape into a fixed streamline model.

3. <u>Transformation of the supersonic paradox into a phase paradox.</u>

Until now we have <u>written</u> $\phi(x,t)$ in the shape of formula (15), in that way creating the image of two waves travelling at the supersonic speed $v$. Moreover these waves are afflicted by a paradoxal cut-off frequency. It is equally possible, however, to arrange wave 1 and wave 2 in the following manner

$$\text{wave 1} \qquad A_1 e^{-\frac{1}{2}mx}\, e^{-jx\left(\frac{\omega}{c} - \sqrt{\frac{\omega^2}{c^2} - \frac{m^2}{4}}\right)}\, e^{j\omega\left(t + \frac{x}{c}\right)} \qquad (60)$$

$$\xleftarrow{\quad c \quad}$$

$$\text{wave 2} \qquad A_2 e^{-\frac{1}{2}mx}\, e^{jx\left(\frac{\omega}{c} - \sqrt{\frac{\omega^2}{c^2} - \frac{m^2}{4}}\right)}\, e^{j\omega\left(t - \frac{x}{c}\right)} \qquad (61),$$

$$\xrightarrow{\quad c \quad}$$

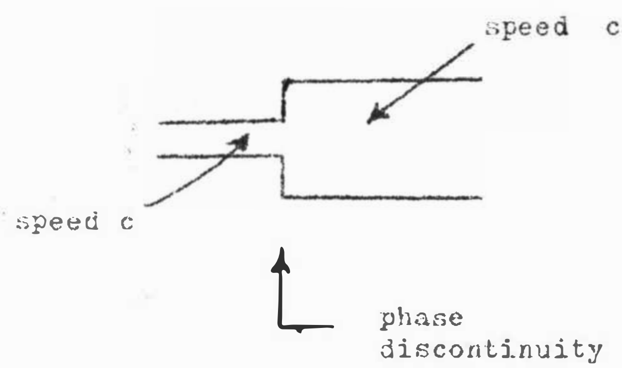
imposing on us the image that the waves travel at the <u>normal velocity of sound $c$</u> but suffer from a strange phase angle

$$\varphi(x,\omega) = x\left(\frac{\omega}{c} - \sqrt{\frac{\omega^2}{c^2} - \frac{m^2}{4}}\right) \qquad (62)$$

that depends on $x$ and $\omega$.

The breath-taking paradox is still present because , as (60) and ( 61) clearly show, below the frequency $\omega_0 = \frac{1}{2} m c$ no wave propagation at the speed c is possible either.

Introduction of the phase paradox does not remove the difficulties : it merely shifts them to another domain. Nevertheless, the phase paradox , allowing the waves to travel at the speed c , gives us a better insight into the nature of , for instance , the twin-tube model depicted in figure 4 .



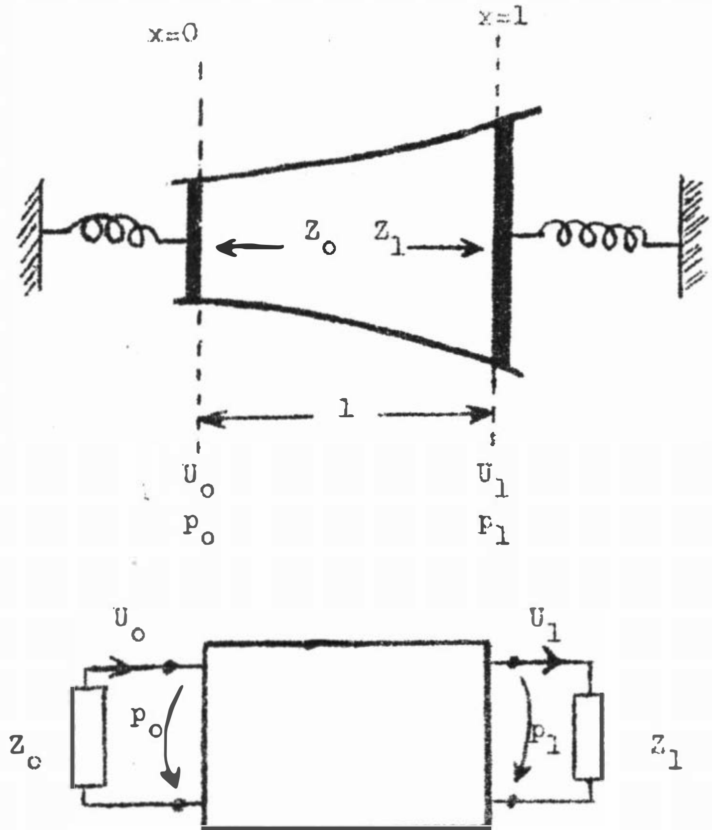

F I G U R E 4

The waves in the twin-tube model

Because the tubes have a constant cross-area , in each of them waves are allowed to travel at the speed c without experiencing phase troubles because, as is apparent from (62) , $\varphi = 0$ for $m = 0$. When we consider the twin-tube as a unit , however, there is a sudden step in the phase at the joint where both tubes meet. This places the twin-tube in the same class as the exponential Webster model , it being understood that the phase troubles in the exponential horn are distributed continuously along its length.

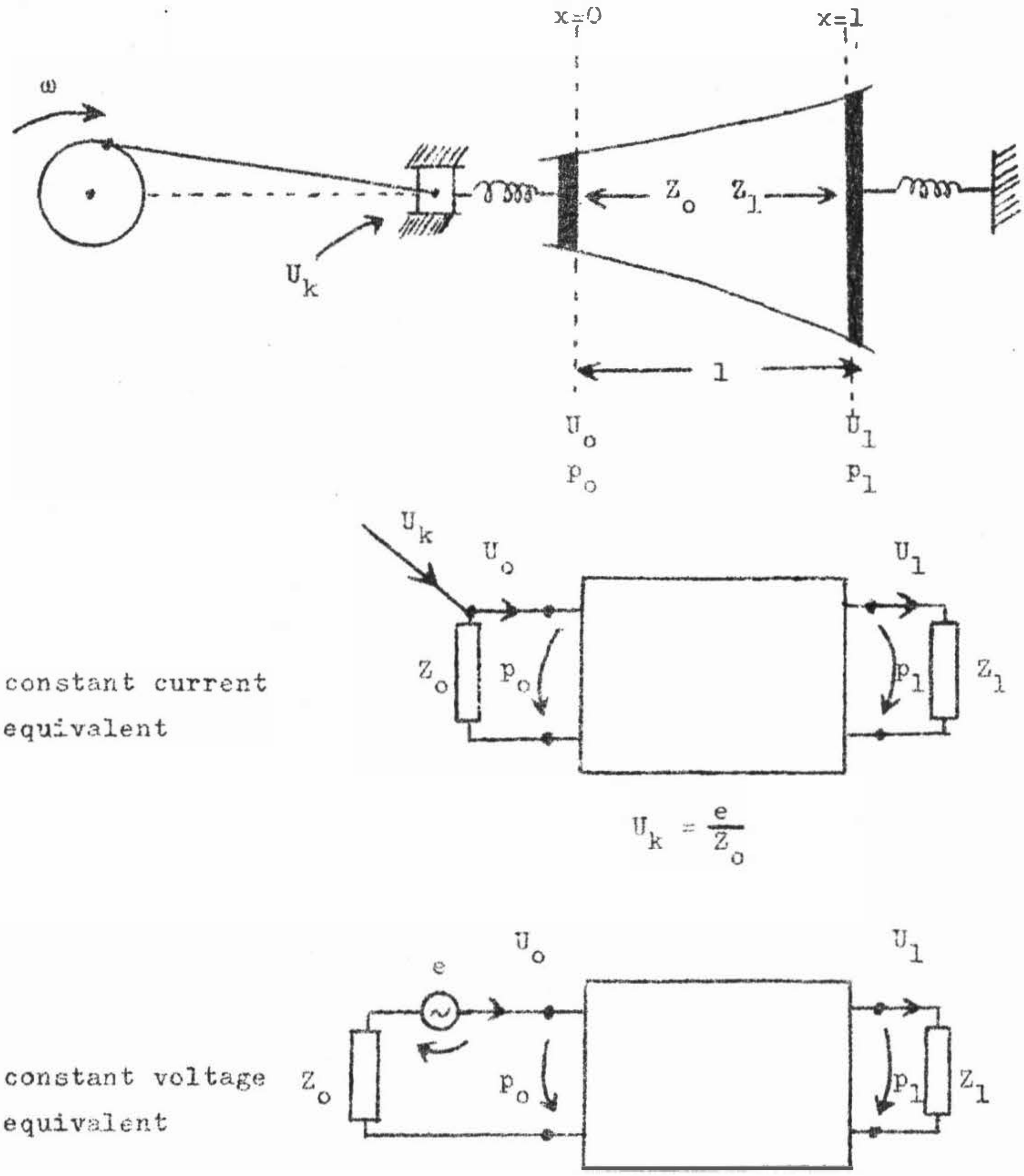
## 4. On the nature of the boundary conditions.

This final paragraph draws the attention to the necessity of clearly defining the boundary conditions when one undertakes to predict the character of the vibrations in a tube that is supposed to obey Webster's ( or anybody's ) horn equation.




F I G U R E 5

T H E P A S S I V E C A S E
(no external excitation) leading
to the eigenvalue method.




F I G U R E 6

T H E T R A N S M I S S I O N C A S E
(external excitation) leading to the
resonance frequencies .

To begin with, in any case the inter-faces with the outer world at the positions $x = 0$ and $x = 1$ must be considered as pistons that move to and fro in the openings, hindered by the external impedances $Z_0$ and $Z_1$.

When we suppose, as is shown in fig. 5, that one limits oneself to the case where the pistons are passively reacting merely to the vibrations <u>inside</u> the tube, we have wilfully manoeuvered ourselves into an eigenvalue problem. We then find that vibrations in the tube are exclusively possible in certain modes, namely, damped oscillations at prescribed, discrete frequencies. In the special case that $Z_0 = \infty$ and $Z_1 = 0$, these oscillations are undamped pure sines, in that way giving rise to a sort of acoustic perpetuum mobile.

When, however, we consider the horn as a means for <u>transmitting</u> energy from a loudspeaker membrane to a listener or from a throat to the mouth opening, we have to calculate the situation depicted in fig. 6.

When we now drive the horn from a sinusoidal source, seen either as a constant velocity fed in parallel or as a constant pressure fed in series with $Z_0$, we are able to calculate, in the stationary state, for instance, the velocity $U_1$ in the mouth opening. We then find that the amplitude of $U_1$ reaches a maximum for certain frequencies, the so-called resonance frequencies. The less damping the system shows, the sharper these resonances turn out to be. In the special case that $Z_0 = \infty$ and $Z_1 = 0$ the resonance frequencies found via the transmission method coincide with the discrete natural frequencies found via the eigenvalue method. Moreover, in this special case the resonances turn out to be so sharp that there is zero amplitude for all frequencies that differ from the resonance frequencies.

The, mathematically speaking, legal transmission method is not to be blamed for the existence of the supersonic paradox: this paradox remains a typical consequence of Webster's equation.