Information in Spoken Language

A quantitative approach

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Introduction

Statistics is the bookkeeping of information

- Language is about communication
- Communication implies a message
- A message is only useful if it is "surprising" to some extend
- That is, the receiver must be uncertain about the content of the message
- Information and probability quantify uncertainty
- Information is the more fundamental concept

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- A measure of the frequency of outcomes
- A measure of chance given what is known
- A number between 0 and 1 (inclusive)
- A measure of our knowledge (or ignorance)
- Boring?

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Introduction: Axioms

Probability: if $E_1, \ldots, E_n$ are possible outcomes of an observation, then $P(E_i)$ is the probability of outcome E_i iff

1. $0 \leq P(E_i) \leq 1$
2. $P(E_1 \vee \cdots \vee E_i \vee \cdots \vee E_n) = 1$
3. Additivity: $P(E_1 \vee E_2) = P(E_1) + P(E_2)$ where E_1 and E_2 are mutually exclusive.
4. Countable additivity: $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$ for $n = 1, 2, \ldots, N$ where $E_1, E_2, \ldots$ are mutually exclusive.

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Introduction: Conventions

Take three observables: *Color*, *Flower*, and *Place*

The following conventions will be used:

1. $P(C = Red)$: the probability of seeing a *Red flower*
2. $P(C = Red, F = Rose)$: the probability of seeing a *Red Rose*
3. $P(C = Red|F = Rose)$: the probability of seeing a *Red Flower*, given that the flower is a *Rose*
4. $P(C = Red, F = Rose|P = Flower\ Shop) \leq 1$
5. $P(Red \vee Blue) = P(Red) + P(Blue)$
   Basic sum rule for probabilities
6. $P(C, F|P) = P(C|F, P) \cdot P(F|P) = P(F|C, P) \cdot P(C|P)$
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Probability distributions

Useful distributions, called Probability Density Functions (pdf)

- Uniform distribution, discrete and uniform
- Poisson distribution
- Normal (Gaussian) distribution
- Zipf distribution
- Mean value, μ, is called Expected value
  $\mu = E[x] = \int_{-\infty}^{+\infty} x \cdot P(x) dx$
- Distribution width is called *Standard Deviation* which is defined as $\sigma = \sqrt{E[(x - E(x))^2]}$

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Probability distributions: Uniform Discrete

N, equally probable and equally spaced values $\{E_1, \ldots, E_n\}$ (possibly if $N \to \infty$)

- Each category, E_i, has the same probability
- $P(E_i) = 1/N$
- Example: Dice $\{1, \ldots, 6\}$ and coins $\{Head, Tail\}$
- Most basic distribution
- Default if only the number of values is known
- Mean $\mu = \frac{1}{N}\sum_{i=1}^{N} E_i = \frac{1}{2}(E_1 + E_N)$
- Variance $\sigma^2 = \frac{1}{12}(E_N - E_1)^2$

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Probability distributions: Uniform Continuous

Equally probable values in interval $[a, b]$

- $Pdf(x) = \dfrac{1}{b - a}$
- Most basic distribution (continuous case)
- Default if only the range is known
- Mean $\mu = \dfrac{a + b}{2}$
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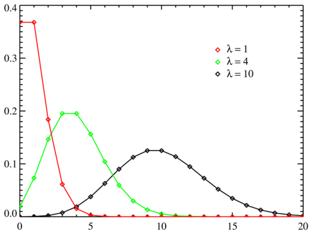

$$Pdf(k; \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

k: count, λ: rate

Rare events occuring with a fixed rate λ

- Mushrooms per meter of forest, typing errors per page, radio-active decay
- Average and variance are identical $\mu = \sigma^2 = \lambda$
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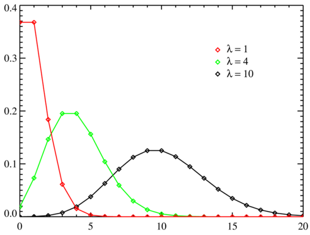
http://en.wikipedia.org/wiki/Poisson_distribution

Probability distributions: Poisson



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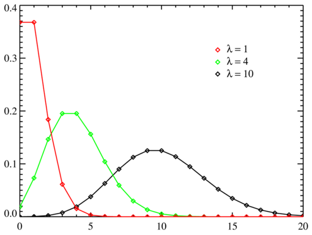

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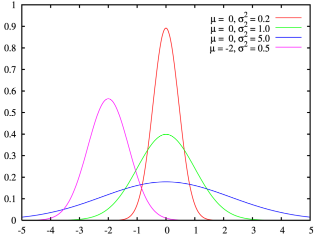


$$Pdf(x;\mu,\sigma) = \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

x: observable

μ: Average

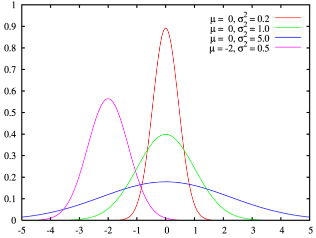
σ^2: variance

General measurements

- Many physical and physiological measurements, counting
- Default if both an average and a variance are known
- A sum of a large number of independent variables is approximately normal (under certain conditions)

http://en.wikipedia.org/wiki/Normal_distribution

Probability distributions: Normal or Gaussian



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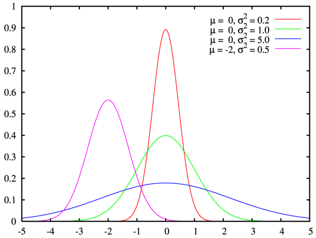

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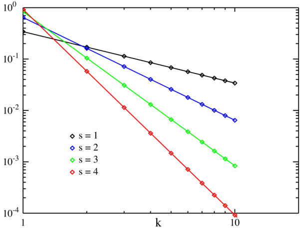

$$Pdf(k; s, N) = \frac{\frac{1}{k^s}}{\sum_{n=1}^{N} \frac{1}{n^s}}$$

k: rank; s: exponent

N: number of elements

note logarithmic scales

Product of frequency and rank is constant: $f_i \approx C \cdot \frac{1}{r_i}$

- Word frequencies, city sizes, high incomes, earthquake sizes
- Default with power laws
- For word frequencies, $s \approx 1$

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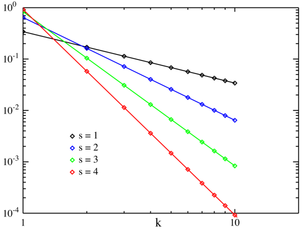

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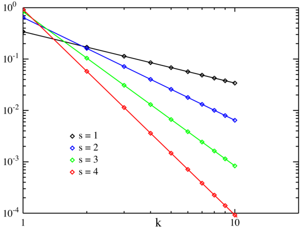

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Product of frequency and rank is constant: $f_i \approx C \cdot \frac{1}{r_i}$

- Word frequencies, city sizes, high incomes, earthquake sizes
- Default with power laws
- For word frequencies, $s \approx 1$

http://en.wikipedia.org/wiki/Zipf_distribution

Bayesian probabilities

Incorporating knowledge

- Probability is predicting outcomes from knowledge
- Explicitely formulate knowledge as probabilities
- Formalize the probability of a hypothesis
- Destinguish a priori (knowledge) and a posteriori (observations) probabilities
- Determine the information content of a single observation

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\begin{aligned}
P(Data, Hypothesis) &= P(Hypothesis|Data) \cdot P(Data) \\
&= P(Data|Hypothesis) \cdot P(Hypothesis) \\
&\Leftrightarrow \\
P(Hypothesis|Data) &= \frac{P(Data|Hypothesis) \cdot P(Hypothesis)}{P(Data)}
\end{aligned}
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Express $P(Hypothesis|Data)$:

- As a function of the measurements
- And the a priori probability of the hypothesis
- Normalized by the a priori probability of the data
- The normalization probability can often be ignored, as it will be identical for all hypotheses

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Bayesian probabilities: Toy example

Where has Watson most likely been: *Market*, *Garden*, *Meadow*, *Park*?

- Watson caries a Yellow *Buttercup*
- He divides his walks equally along these "*places*" (uniform prior)
- Which is most likely, obtaining a Yellow *Buttercup* in a *Market*, a *Garden*, a *Meadow*, or a *Park*?
- In formula:

$$\underset{p}{argmax} P(p|Y,B) = \underset{p}{argmax} P(Y,B|p) \cdot P(p)$$

$$where\ \ p \in \{Market, Garden, Meadow, Park\}$$

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Information and probabilities: Surprise!

Information is a quantification of *surprise*

- Information depends on probability p_i
- A more surprising observation, ie, a lower p_i, caries more information
- Information should be additive, two CD's can carry twice the information of one CD
- Define information in observation O_i with probability p_i as $h(p_i) = -\log_2 p_i$

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Information and probabilities: Uncertainty

The uncertainty is the average information content and is called *Entropy*, $H(p_1, p_2, \ldots, p_n)$. *Entropy* should be:

- Independent of the labeling, ie, numbering, of p_i
- Decomposable, splitting a category in two gives: $H'(p'_1, p''_1, \ldots) = H(p_1, \ldots) + p_1 \cdot H(\frac{p'_1}{p_1}, \frac{p''_1}{p_1})$
- Continuous, a small change in the probabilities should result in a small change in *entropy*
- Monotonic, for a uniform distibution of n items, entropy increases monotonically with the number of categories $n \geq 1$
- $\Rightarrow H(p_1, p_2, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2(p_i)$

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Information and probabilities

Probability distributions have *entropies*: Examples

- Discrete Uniform distribution: $H(\frac{1}{N}) = \log_2(N)$
- Continuous Uniform distribution $[a, b]$: $H(\frac{1}{b-a}) = \log_2(b-a)$
- Poisson distribution: $H(k; \lambda) = \lambda[\frac{1}{\ln(2)} - \log_2(\lambda)] + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k \log_2(k!)}{k!}$
- Normal (Gaussian) distribution: $H(x; \mu, \sigma) = \log_2(\sigma\sqrt{2\pi e})$
- Zipf distribution ($C_{N,s} = \sum_{k=1}^{N} \frac{1}{k^s}$): $H(k; s, N) = \frac{s}{C_{N,s}} \sum_{k=1}^{N} \frac{\log_2(k)}{k^s} + \log_2(C_{N,s})$

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Information and probabilities: Measuring

Information is the reduction of uncertainty

$$\underset{parameter}{X} \overset{channel}{\Rightarrow} \underset{observation}{Y}$$

- Entropy in X before the observation: $H(X)$
- Entropy after the observation of value of Y: $H(X|Y)$
- Average information gained through observing Y: $I(X|Y) = H(X) - H(X|Y)$
- If there is *no* uncertainty left after observing Y, ie, $H(X|Y) = 0$: $I(X|Y) = H(X)$
- If X and Y are independent, ie, $H(X|Y) = H(X)$, then $I(X|Y) = 0$
- Always, $H(X|Y) \leq H(X) \Rightarrow I(X|Y) \leq H(X)$
- It is common to use $H(\cdot)$ as a synonym of $I(\cdot)$

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$$KL(p:q) = \sum_i p_i \log_2 \frac{p_i}{q_i} \quad \bigvee \quad \int_{-\infty}^{\infty} p(x) \log_2 \frac{p(x)}{q(x)} dx$$

discontinuous *continuous*

$H(p,q) = \sum_i p_i \log_2 q_i$: Cross Entropy

Kullback-Leibler distance

- A non-symmetric divergence: $KL(p:q) \neq KL(q:p)$
- Measures "distance" between prob. distributions
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- Example: Word distributions as a distance between document types

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Compression: Minimum size

Entropy, $H(A)$ can be understood as the minimal number of bits needed to fully *specify* A given a known production process

- In an unknown process, $K(A)$ replaces $H(A)$ as the information content
- $K(A)$: Minimum number of bits to reconstruct A
- $K(A)$ is the theoretical lower limit of compression size $C(A)$
- Practical (lossless) compression packages, $C(A)$, eg, ZIP, GZIP, BZIP2 etc. never reach this limit

$K(A)$ is called the Kolmogorov complexity [Vitanyi(2005)][Chater and Vitanyi(2001)]

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- $K(A)$: Minimum number of bits to *reconstruct* A
- $K(A)$ is the theoretical lower limit of *compression size* $C(A)$
- Practical (lossless) compression packages, $C(A)$, eg, ZIP, GZIP, BZIP2 etc. never reach this limit

$K(A)$ is called the Kolmogorov complexity [Vitanyi(2005)][Chater and Vitanyi(2001)]

Compression: Similarity metric

$$
\begin{aligned}
NCD(A,B) &= \frac{min\{C(A|B), C(B|A)\}}{max\{C(A), C(B)\}} \\
&= \frac{C(AB) - min\{C(A), C(B)\}}{max\{C(A), C(B)\}}
\end{aligned}
$$

NCD: Normalized Compression Distance

Similarity by compression

- Always $C(AB) \leq C(A) + C(B)$ $(+constant)$
- Estimate entropy by suitable "long range" compression
- $K(text) \leq C(text)$ in bits

`http://www.complearn.org/ncd.html`

[Chen et al.(2004)Chen, Li, Ma, and Vitányi][Vitanyi(2005)][Chater and Vitanyi(2001)]

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Markov Chains

Words and letters never follow each other at random

- The simplest language "model" predicts the next word based on the previous word
- Markov chain: $P(w_{i+1}|w_i) = \dfrac{P(w_{i+1}, w_i)}{P(w_i)}$
- Can be extended to more words
- Large amounts of text are needed to determine $P(w_{i+1}, w_i)$ reliably
- Example Markov text:
  Step which one could go be grabbed. People to Do that my the former Netscape brand's fortunes that means indent command to The user visible displays a.

http://en.wikipedia.org/wiki/Markov_chain

Generate texts: http://www.jwz.org/dadadodo/

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Markov Chains: Language models

With Markov chains, or N-grams, the probability of a sequence can be calculated

- What is the probability of encounting a sentence $(w_1, \ldots, w_n)$?
- A human style language model is not known
- Use *N-gram* Markov chains
- $P(w_1, \ldots, w_n) = \prod_{i=1}^{n} P(w_i | w_1, \ldots, wi - 1)$ (exact)
- $P(w_1, \ldots, w_n) \approx \prod_{i=1}^{n} P(w_i | w_{i-N+1}, \ldots, w_{i-1})$ (*N-gram* approximation)

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Markov Chains: Perplexity

$$P_X(Model) = 2^{H_X(w_i|W_{1\ldots i-1})}$$
$$H_X(w_i|W_{1\ldots i-1}) = -\sum_{\{W\}} P_{observed}(w_i|\ldots)\log P_{model}(w_i|\ldots)$$

$H_X(\cdot)$: Cross Entropy

Perplexity: "average" number of choices for the next word

- Matches observed with modelled word order
- A better language model has a lower perplexity
- For an *N-gram* Markov chain the perplexity is well defined
- Using the model entropy io. the cross entropy estimates the quality of the model on the training corpus

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Maximum Entropy

$$
\begin{aligned}
findp^* &= \underset{p \in C}{argmax} H(p) \\
&= \underset{p \in C}{argmax} \left(-\sum_{x,y} \tilde{p}(x) p(y|x) \log p(y|x) \right)
\end{aligned}
$$

Which model, p^*, fits my data best and by what criterium?

- Quantify all constraints (knowledge) and determine the set of possible distributions $p \in C$
- Determine the average entropy, $H(y|x)$, over the observed (measured) probabilities $\tilde{p}(x)$
- The best distribution, p^* has the highest entropy

[Berger()][Berger(1996)] [Berger et al.(1996)Berger, della Pietra, and della Pietra] [Maxent()]

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Maximum Entropy: Kangaroo example

$\frac{1}{3}$ of all kangaroos have blue eyes and $\frac{1}{3}$ are left handed

blue eyed	Left Handed true	Left Handed false	tot
true	x	$\frac{1}{3} - x$	$\frac{1}{3}$
false	$\frac{1}{3} - x$	$\frac{1}{3} + x$	$\frac{2}{3}$
tot	$\frac{1}{3}$	$\frac{2}{3}$	1

How many are both blue eyed and left handed?

- All $0 \leq x \leq \frac{1}{3}$ are possible
- $H(x = \frac{1}{9}) \approx 1.84$ has maximum entropy
- $x = \frac{1}{9}$ is the only solution with uncorrelated eye color and handedness

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Bibliography V

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End of Terms and Conditions

Bibliography IX

Appendix: How to Apply These Terms to Your New Programs

If you develop a new program, and you want it to be of the greatest possible use to the public, the best way to achieve this is to make it free software which everyone can redistribute and change under these terms.
To do so, attach the following notices to the program. It is safest to attach them to the start of each source file to most effectively convey the exclusion of warranty; and each file should have at least the "copyright" line and a pointer to where the full notice is found.

> *one line to give the program's name and a brief idea of what it does.*
> *Copyright (C) yyyy name of author*
> *This program is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation; either version 2 of the License, or (at your option) any later version.*
>
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> *You should have received a copy of the GNU General Public License along with this program; if not, write to the Free Software Foundation, Inc., 51 Franklin Street, Fifth Floor, Boston, MA 02110-1301, USA.*

Also add information on how to contact you by electronic and paper mail.
If the program is interactive, make it output a short notice like this when it starts in an interactive mode:

> *Gnomovision version 69, Copyright (C) yyyy name of author*
> *Gnomovision comes with ABSOLUTELY NO WARRANTY; for details type 'show w'.*
> *This is free software, and you are welcome to redistribute it under certain conditions; type 'show c' for details.*

Bibliography X

The hypothetical commands `show w` and `show c` should show the appropriate parts of the General Public License. Of course, the commands you use may be called something other than `show w` and `show c`; they could even be mouse-clicks or menu items—whatever suits your program.
You should also get your employer (if you work as a programmer) or your school, if any, to sign a "copyright disclaimer" for the program, if necessary. Here is a sample; alter the names:

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> *signature of Ty Coon, 1 April 1989*
> *Ty Coon, President of Vice*

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