Information in Spoken Language
A quantitative approach

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LOT winterschool 2006

Amsterdam Center for Language and Communication (ACLC)
1 Introduction to Information Theory

- Introduction
- Probability distributions
- Bayesian probabilities
- Information and probabilities
- Relative entropy
- Compression
- Markov Chains
- Maximum Entropy
- Bibliography
Introduction

Statistics is the bookkeeping of information

- Language is about communication
- Communication implies a message
- A message is only useful if it is “surprising” to some extend
- That is, the receiver must be uncertain about the content of the message
- Information and probability quantify uncertainty
- Information is the more fundamental concept
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- A measure of the frequency of outcomes
- A measure of chance given what is known
- A number between 0 and 1 (inclusive)
- A measure of our knowledge (or ignorance)
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Introduction: Axioms

Probability: if \( E_1, \ldots, E_n \) are possible outcomes of an observation, then \( P(E_i) \) is the probability of outcome \( E_i \) iff

1. \( 0 \leq P(E_i) \leq 1 \)
2. \( P(E_1 \lor \cdots \lor E_i \lor \cdots \lor E_n) = 1 \)
3. Additivity: \( P(E_1 \lor E_2) = P(E_1) + P(E_2) \)
   where \( E_1 \) and \( E_2 \) are mutually exclusive.
4. Countable additivity:
   \[ P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) \] for \( n = 1, 2, \ldots, N \)
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Introduction: Conventions

Take three observables: Color, Flower, and Place

The following conventions will be used:

1. \( P(C = \text{Red}) \): the probability of seeing a Red flower
2. \( P(C = \text{Red}, F = \text{Rose}) \): the probability of seeing a Red Rose
3. \( P(C = \text{Red}|F = \text{Rose}) \): the probability of seeing a Red Flower, given that the flower is a Rose
4. \( P(C = \text{Red}, F = \text{Rose}|P = \text{Flower Shop}) \leq 1 \)
5. \( P(\text{Red} \lor \text{Blue}) = P(\text{Red}) + P(\text{Blue}) \)
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6. \( P(C, F|P) = P(C|F, P) \cdot P(F|P) = P(F|C, P) \cdot P(C|P) \)
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Probability distributions

Useful distributions, called Probability Density Functions (pdf)

- Uniform distribution, discrete and uniform
- Poisson distribution
- Normal (Gaussian) distribution
- Zipf distribution
- Mean value, \( \mu \), is called Expected value
  \[ \mu = E[x] = \int_{-\infty}^{+\infty} x \cdot P(x)dx \]
- Distribution width is called Standard Deviation which is defined as
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Probability distributions: Uniform Discrete

\(N\), equally probable and equally spaced values \(\{E_1, \ldots, E_n\}\) (possibly if \(N \to \infty\))

- Each category, \(E_i\), has the same probability
  - \(P(E_i) = \frac{1}{N}\)
  - Example: Dice \(\{1, \ldots, 6\}\) and coins \(\{\text{Head}, \text{Tail}\}\)
  - Most basic distribution
  - Default if only the number of values is known
  - Mean \(\mu = \frac{1}{N} \sum_{i=1}^{N} E_i = \frac{1}{2}(E_1 + E_N)\)
  - Variance \(\sigma^2 = \frac{1}{12}(E_N - E_1)^2\)

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Equally probable values in interval \([a, b]\)

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Probability distributions: Poisson

Pdfs(k; \lambda) = \frac{e^{-\lambda} \lambda^k}{k!}

k: count, \lambda: rate

Rare events occurring with a fixed rate \lambda

- Mushrooms per meter of forest, typing errors per page, radio-active decay
- Average and variance are identical \mu = \sigma^2 = \lambda
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Probability distributions: Normal or Gaussian

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\[ \text{Pdf}(x; \mu, \sigma) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma \sqrt{2\pi}} \]

- \( x \): observable
- \( \mu \): Average
- \( \sigma^2 \): variance

General measurements

- Many physical and physiological measurements, counting
  - Default if both an average and a variance are known
  - A sum of a large number of independent variables is approximately normal (under certain conditions)

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Probability distributions: Zipf

\[Pdf(k; s, N) = \frac{1}{k^s} \sum_{n=1}^{N} \frac{1}{n^s}\]

- \(k\): rank; \(s\): exponent
- \(N\): number of elements
- note logarithmic scales

Product of frequency and rank is constant: \(f_i \approx C \cdot \frac{1}{r_i}\)

- Word frequencies, city sizes, high incomes, earthquake sizes
- Default with power laws
- For word frequencies, \(s \approx 1\)


[Dover (2004)]
[Kawamura and Hatano (2002)]
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\[ P(\text{Data, Hypothesis}) = P(\text{Hypothesis|Data}) \cdot P(\text{Data}) \]

\[ = P(\text{Data|Hypothesis}) \cdot P(\text{Hypothesis}) \]

\[ \Leftrightarrow \]

\[ P(\text{Hypothesis|Data}) = \frac{P(\text{Data|Hypothesis}) \cdot P(\text{Hypothesis})}{P(\text{Data})} \]

Express \( P(\text{Hypothesis|Data}) \):

- As a function of the measurements
- And the a priori probability of the hypothesis
- Normalized by the a priori probability of the data
- The normalization probability can often be ignored, as it will be identical for all hypotheses
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\]

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\iff
\]

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Bayesian probabilities: Toy example

Where has Watson most likely been: Market, Garden, Meadow, Park?

- Watson carries a Yellow Buttercup
- He divides his walks equally along these “places” (uniform prior)
- Which is most likely, obtaining a Yellow Buttercup in a Market, a Garden, a Meadow, or a Park?
- In formula:

\[
\arg\max_P P(p | Y, B) = \arg\max_P P(Y, B | p) \cdot P(p)
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Information and probabilities: Surprise!

Information is a quantification of *surprise*

- Information depends on probability $p_i$
- A more surprising observation, ie, a lower $p_i$, carries more information
- Information should be additive, two CD’s can carry twice the information of one CD
- Define information in observation $O_i$ with probability $p_i$ as $h(p_i) = - \log_2 p_i$
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Information and probabilities: Uncertainty

The uncertainty is the average information content and is called *Entropy*, $H(p_1, p_2, \ldots, p_n)$. *Entropy* should be:

- **Independent** of the labeling, ie, numbering, of $p_i$
- Decomposable, splitting a category in two gives:
  \[ H'(p'_1, p''_1, \ldots) = H(p_1, \ldots) + p_1 \cdot H\left(\frac{p'_1}{p_1}, \frac{p''_1}{p_1}\right) \]
- Continuous, a small change in the probabilities should result in a small change in *entropy*
- Monotonic, for a uniform distribution of $n$ items, entropy increases monotonically with the number of categories $n \geq 1$

\[ \Rightarrow H(p_1, p_2, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2(p_i) \]

See chapter 1 of [Bavaud et al.(2005)Bavaud, Chappelier, and Kohlas]
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Probability distributions have *entropies*: Examples

- **Discrete Uniform distribution**: \( H\left(\frac{1}{N}\right) = \log_2(N) \)

- **Continuous Uniform distribution** \([a, b] \): \( H\left(\frac{1}{b-a}\right) = \log_2(b - a) \)

- **Poisson distribution**: 
  \[ H(k; \lambda) = \lambda\left[\frac{1}{\ln(2)} - \log_2(\lambda)\right] + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k \log_2(k!)}{k!} \]

- **Normal (Gaussian) distribution**:  
  \[ H(x; \mu, \sigma) = \log_2(\sigma \sqrt{2\pi e}) \]

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Information and probabilities

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Information and probabilities: Measuring

Information is the reduction of uncertainty

\[ \begin{align*}
\text{parameter} & \xrightarrow{\text{channel}} \text{observation} \\
X & \Rightarrow Y
\end{align*} \]

- Entropy in \( X \) before the observation: \( H(X) \)
- Entropy after the observation of value of \( Y \): \( H(X|Y) \)
- Average information gained through observing \( Y \):
  \[ I(X|Y) = H(X) - H(X|Y) \]
- If there is no uncertainty left after observing \( Y \), ie, \( H(X|Y) = 0 \):
  \[ I(X|Y) = H(X) \]
- If \( X \) and \( Y \) are independent, ie, \( H(X|Y) = H(X) \), then
  \[ I(X|Y) = 0 \]
- Always, \( H(X|Y) \leq H(X) \Rightarrow I(X|Y) \leq H(X) \)
- It is common to use \( H(\cdot) \) as a synonym of \( I(\cdot) \)

See chapter 1 of [Bavaud et al.(2005)Bavaud, Chappelier, and Kohlas]
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Relative entropy: $K(p : q) = H(p, q) - H(p)$

$$K(p : q) = \sum_i p_i \log_2 \frac{p_i}{q_i} \text{ discontinuous} \quad \sqrt{\int_{-\infty}^{\infty} p(x) \log_2 \frac{p(x)}{q(x)} \, dx \text{ continuous}}$$

$$H(p, q) = \sum_i p_i \log_2 q_i: \text{ Cross Entropy}$$

**Kullback-Leibler distance**

- A non-symmetric divergence: $K(p : q) \neq K(q : p)$
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- Example: Word distributions as a distance between document types

http://en.wikipedia.org/wiki/Kullback-Leibler_distance
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Discontinuous \hspace{1cm} Continuous

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Entropy, $H(A)$ can be understood as the minimal number of bits needed to fully *specify* $A$ given a known production process.

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- $K(A)$: Minimum number of bits to reconstruct $A$.
- $K(A)$ is the theoretical lower limit of compression size $C(A)$.
- Practical (lossless) compression packages, $C(A)$, eg, ZIP, GZIP, BZIP2 etc. never reach this limit.

$K(A)$ is called the Kolmogorov complexity [Vitanyi(2005)][Chater and Vitanyi(2001)].
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Practical (lossless) compression packages, $C(A)$, eg, ZIP, GZIP, BZIP2 etc. never reach this limit.

$K(A)$ is called the Kolmogorov complexity [Vitani(2005)][Chater and Vitani(2001)].
Entropy, $H(A)$ can be understood as the minimal number of bits needed to fully specify $A$ given a known production process.

- In an unknown process, $K(A)$ replaces $H(A)$ as the information content.
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$K(A)$ is called the Kolmogorov complexity [Vitanyi(2005)],[Chater and Vitanyi(2001)].
Compression: Similarity metric

\[
NCD(A, B) = \frac{\min \{C(A|B), C(B|A)\}}{\max \{C(A), C(B)\}} = \frac{C(AB) - \min \{C(A), C(B)\}}{\max \{C(A), C(B)\}}
\]

NCD: Normalized Compression Distance

**Similarity by compression**

- Always \(C(AB) \leq C(A) + C(B)\) (+constant)
- Estimate entropy by suitable “long range” compression
- \(K(\text{text}) \leq C(\text{text})\) in bits

http://www.complearn.org/ncd.html

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Words and letters never follow each other at random

- The simplest language “model” predicts the next word based on the previous word
  
  Markov chain: \( P(w_{i+1}|w_i) = \frac{P(w_{i+1}, w_i)}{P(w_i)} \)

- Can be extended to more words
- Large amounts of text are needed to determine \( P(w_{i+1}, w_i) \) reliably
- Example Markov text:
  
  Step which one could go be grabbed. People to Do that my the former Netscape brand’s fortunes that means indent command to The user visible displays a.

http://en.wikipedia.org/wiki/Markov_chain

Generate texts: http://www.jwz.org/dadadodo/
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With Markov chains, or N-grams, the probability of a sequence can be calculated

- What is the probability of encountering a sentence \((w_1, \ldots, w_n)\)?
- A human style language model is not known
- Use N-gram Markov chains
- \(P(w_1, \ldots, w_n) = \prod_{i=1}^{n} P(w_i|w_1, \ldots, w_i - 1)\) (exact)
- \(P(w_1, \ldots, w_n) \approx \prod_{i=1}^{n} P(w_i|w_{i-N+1}, \ldots, w_{i-1})\) (N-gram approximation)
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Markov Chains: Perplexity

\[ P_X(\text{Model}) = 2^{H_X(w_i|W_1...i-1)} \]

\[ H_X(w_i|W_1...i-1) = -\sum_{\{W\}} P_{\text{observed}}(w_i|...) \log P_{\text{model}}(w_i|...) \]

\( H_X(\cdot) \): Cross Entropy

Perplexity: “average” number of choices for the next word

- Matches observed with modelled word order
- A better language model has a lower perplexity
- For an \( N \)-gram Markov chain the perplexity is well defined
- Using the model entropy io. the cross entropy estimates the quality of the model on the training corpus
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Maximum Entropy

\[
\text{find } p^* = \arg\max_{p \in C} H(p) \\
= \arg\max_{p \in C} \left( -\sum_{x,y} \tilde{p}(x)p(y|x) \log p(y|x) \right)
\]

Which model, \( p^* \), fits my data best and by what criterium?

- Quantify all constraints (knowledge) and determine the set of possible distributions \( p \in C \)
- Determine the average entropy, \( H(y|x) \), over the observed (measured) probabilities \( \tilde{p}(x) \)
- The best distribution, \( p^* \), has the highest entropy


[Roni Rosenfeld(1996)]
Maximum Entropy

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Which model, $p^*$, fits my data best and by what criterium?

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[ Berger(1996) ] [ Berger et al. (1996) ] [ Berger, della Pietra, and della Pietra ] [ Maxent() ]
[ Roni Rosenfeld (1996) ]
Maximum Entropy: Kangaroo example

\( \frac{1}{3} \) of all kangaroos have blue eyes and \( \frac{1}{3} \) are left handed

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Handed</th>
<th>tot</th>
</tr>
</thead>
<tbody>
<tr>
<td>blue</td>
<td>( x )</td>
<td>( \frac{1}{3} - x )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>eyed</td>
<td>true</td>
<td>false</td>
<td></td>
</tr>
<tr>
<td>true</td>
<td>( \frac{1}{3} - x )</td>
<td>( \frac{1}{3} + x )</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
<td>false</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tot</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{2}{3} )</td>
<td>1</td>
</tr>
</tbody>
</table>

How many are both blue eyed and left handed?

- All \( 0 \leq x \leq \frac{1}{3} \) are possible
- \( H(x = \frac{1}{9}) \approx 1.84 \) has maximum entropy
- \( x = \frac{1}{9} \) is the only solution with uncorrelated eye color and handedness
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Appendix: How to Apply These Terms to Your New Programs

If you develop a new program, and you want it to be of the greatest possible use to the public, the best way to achieve this is to make it free software which everyone can redistribute and change under these terms. To do so, attach the following notices to the program. It is safest to attach them to the start of each source file to most effectively convey the exclusion of warranty; and each file should have at least the “copyright” line and a pointer to where the full notice is found.

one line to give the program’s name and a brief idea of what it does.
Copyright (C) yyyy name of author
This program is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation; either version 2 of the License, or (at your option) any later version.

This program is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the GNU General Public License for more details.

You should have received a copy of the GNU General Public License along with this program; if not, write to the Free Software Foundation, Inc., 51 Franklin Street, Fifth Floor, Boston, MA 02110-1301, USA.

Also add information on how to contact you by electronic and paper mail.
If the program is interactive, make it output a short notice like this when it starts in an interactive mode:

Gnomovision version 69, Copyright (C) yyyy name of author
Gnomovision comes with ABSOLUTELY NO WARRANTY; for details type ‘show w’.
This is free software, and you are welcome to redistribute it under certain conditions; type ‘show c’ for details.
The hypothetical commands show w and show c should show the appropriate parts of the General Public License. Of course, the commands you use may be called something other than show w and show c; they could even be mouse-clicks or menu items—whatever suits your program.
You should also get your employer (if you work as a programmer) or your school, if any, to sign a “copyright disclaimer” for the program, if necessary. Here is a sample; alter the names:

Yoyodyne, Inc., hereby disclaims all copyright interest in the program
‘Gnomovision’ (which makes passes at compilers) written by James Hacker.
signature of Ty Coon, 1 April 1989
Ty Coon, President of Vice

This General Public License does not permit incorporating your program into proprietary programs. If your program is a subroutine library, you may consider it more useful to permit linking proprietary applications with the library. If this is what you want to do, use the GNU Library General Public License instead of this License.