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COCHLEAR PHENOMENOLOGY

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Abstract

The purpose of this work is to determine filter characteristics for an arbitrary point of the basilar membrane. We will do this in a phenomenological way. Firstly, attention is paid to the equation of motion for the basilar membrane. It appears that the pressure in this equation has to fulfil a special boundary condition. This condition has the shape of a homogeneous mixed type boundary condition for the pressure in the surrounding fluid. We consider this condition as an equation at the real axis of the complex plane. In consequence of this, the solution can be given in an explicit way. Secondly, both the filter characteristics for an arbitrary point of the membrane and the corresponding impulse response are determined. The impulse response is a chirp-like signal. This corresponds to recent physiological observations. We changed the slopes in the amplitude characteristic of the filter. The result is that small variations of these slopes lead to responses which resemble the observed responses.

1. Introduction

Until the observations of Rhode (1971) or Rhode and Robles (1973), the common opinion on the motion of the basilar membrane as a result of pure tone stimulation, was the concept of a travelling wave along the membrane. According to that notion the motion of the membrane resembles a progressive wave which travels from the stapes to the helicotrema. During its travel the wave reaches a maximum in a small region of the membrane around the point of resonance. After that region the amplitudes of the wave rapidly diminishes; however the wave still travels. The concept of a travelling wave was introduced by von Békésy (1928) after his early model observations. Later model studies and observations in preparations of the cochlea (von Békésy, 1960) seemed to support this notion.

Rhode showed that the concept of a travelling wave holds true as far as the point of resonance. After that point amplitudes of the membrane motion are almost negligibly small. Besides, after resonance the successive points of the membrane perform a motion in almost the same phase.

In consequence of von Békésy's observations it is not surprising that even in early attempts to describe the motion of the basilar membrane the intention was present to model the pressure in the surrounding fluid, even beyond resonance, as a travelling wave.

Ranke (1931, 1942) was one of the first who introduced a travelling wave concept in a two dimensional model study. He noticed that the pressure in the cochlear fluid has to obey Laplace's equation and pointed to the fact that the general solution of this equation can be written as the sum of two arbitrary functions, each of which depends on one of the complex conjugate co-ordinates z and $\bar{z}$. Essentially, the shape of Ranke's solution is an expression of the kind $\exp(c(x+iy))$ at the membrane axis $y=0$ of the complex plane. The constant c is a complex quantity. His idea was to fit this constant so that for successive points of the membrane the pressure and the velocity at the membrane obey the definition of the local impedance. From his work follows that just after resonance the amplitude of the pressure strongly diminishes and that near resonance the local wavelengths are relatively short.

Siebert (1974) re-investigated Ranke's question. He proposed a solution for the pressure in a two dimensional box-like model of the cochlea. Under the so called short wave assumption he arrived at solutions for the motion of the basilar membrane. The relevant part of the solutions shows a wave which travels at different sides of the point of resonance in opposite directions but always towards resonance. Apart from technical details of his analysis, the numerical implementation of these waves points to singular behaviour near resonance.

In a review article on cochlear models Schroeder (1975) expresses his dissatisfaction on the short wave approximations with the sentence "This kind of modelling is out". Fortunately, de Boer (1979, 1984) studied the short wave case again. In addition to an improvement of Siebert's mathematical description, he pointed to the physical phenomenon that the point of resonance at the membrane locally acts as a 'sink' for the energy which is present both at the membrane and in its fluid-like environment. This is an indication that the influence of resonance at the membrane not only determines what happens at the membrane but dictates what happens in the surrounding fluid too. This corresponds to classical ideas from complex analysis that functional behaviour is determined by the presence of singularities. There is no reason at all to go away from those ideas and, what is more, some phase characteristics in recent observations (Ruggero et. al., 1997) seem to underline that indeed near resonance this effect is present in the motion of the basilar membrane.

Near resonance behaviour takes place at and near the membrane. Then the question arises: is it really necessary to introduce boundary value problems for the whole cochlea in order to find this behaviour? The rhetorical character of this sentence includes the answer. In this chapter we will elucidate this.

Here we again start with the observation that it is sufficient to determine the hydrodynamic pressure at the membrane. Once the pressure is known, the motion of the membrane readily follows.

Let us consider a general shape for an oscillating pressure wave at the membrane. This wave can be written as $p(x,t)=\hat{p}(x,\omega)\cos(\omega t+\varphi(x,\omega))$. The amplitude of the wave is $\hat{p}(x,\omega)$ and $\varphi(x,\omega)$ is the phase function. ω is the frequency of the oscillations and x the length parameter along the membrane. Any undulatory behaviour follows from $\varphi(x,\omega)$. When the slope of this function is negative the wave travels to the right. When the slope is positive the wave travels in the opposite direction.

As has been noted by Ranke, the general solution of Laplace's equation can be written as the sum of two functions. Each of these functions depends on one of the complex conjugate co-ordinates z and $\bar{z}$. Therefore, we shall conceive the travelling wave which obeys Laplace's equation at the membrane as a limiting function of two complex conjugate functions $p_1(\bar{z},t)$ and $p_2(z,t)$ so that

$$2p(x,t) = \lim_{y\to 0}\left(p_1(\bar{z},t) + p_2(z,t)\right) \ , \tag{1}$$

in which

$$\lim_{y\to 0} p_1(\bar{z},t) = \hat{p}(x,\omega)\exp\left(+i\left(\omega t + \varphi(x,\omega)\right)\right)$$

and

$$\lim_{y\to 0} p_2(z,t) = \hat{p}(x,\omega)\exp\left(-i\left(\omega t + \varphi(x,\omega)\right)\right) \ .$$

In (1) the function $p_1(\bar{z},t)$ depends on positive frequencies and $p_2(z,t)$ on negative ones. Therefore, we may expect that $p_2(z,t)$ and $p_1(\bar{z},t)$ are complex conjugate. It will appear that, in order to obey this property, we have to distinguish between the upper plane approximation $z = x + i0$ for $p_2(z,t)$ and the lower plane approximation $\bar{z} = x - i0$ for $p_1(\bar{z},t)$. In this article we will distinguish between both approximations.

At first we will shortly derive the key of the problem, namely the basilar membrane condition as an equation for the pressure. Then, the solution in the place domain will be discussed. After that, filter functions of an arbitrarily point of the membrane will be discussed and compared with some experimental observations.

2. A discontinuity as far as resonance

In this section we shall pay attention to properties of the basilar membrane which directly follow from the point of resonance as a mathematical singularity. Our starting point is the linear equation of motion

$$\frac{\partial^2 u_{mn}}{\partial t^2} + \omega_o^2 u_{mn} = -\frac{2}{m}p \ . \tag{2}$$

Here, u_{mn} is the deflection normal to the membrane and $-2p$ is the pressure difference across the membranous strip and m the mass per unit of area. The normal at the membrane is denoted by n. At the membrane u_{mn} coincides with the normal component u_n of the fluid deflection and the pressure there equals the fluid pressure.

Then, both quantities have to obey the same Euler equation normal to the membrane. Therefore, in absence of additional forces, we have in the linear case the additional requirement

$$\frac{\partial^2 u_{mn}}{\partial t^2} = -\frac{1}{\rho}\frac{\partial p}{\partial n} \quad . \tag{3}$$

The difference between (2) and (3) reads

$$\frac{1}{\rho}\frac{\partial p}{\partial n} - \frac{2}{m} p = \omega_0^2 u_{mn} \quad . \tag{4}$$

and has the shape of an inhomogenous radiation condition. Because the membrane deflection equals the fluid deflection normal to the membrane it is superfluous to use special indices. For that reason the indices $_m$ and $_{mn}$ will be omitted.

The present equations are linear. Then it is attractive to apply the technique of Laplace transforms. For the sake of convenience, we shall assume that we only deal with existing transforms (Spiegel, 1965). Moreover, we assume that all initial conditions are zero. In that case the transform of equation (2) is

$$\left(s^2 + \omega_0^2\right)\bar{u} = -\frac{2}{m}\bar{p} \quad , \tag{5}$$

in which $\bar{u}$ and $\bar{p}$ are the transformed deflection and pressure respectively. The transforms of (3) and (4) are

$$s^2\bar{u} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial n} \quad ,$$

and

$$\frac{1}{\rho}\frac{\partial \bar{p}}{\partial n} - \frac{2}{m}\bar{p} = \omega_0^2 \bar{u} \quad ,$$

respectively.

The deflection $\bar{u}$ can be eliminated from the last two equations. This leads to the equation for the pressure in the shape

$$\left(1 + \frac{\omega_0^2}{s^2}\right)\frac{\partial \bar{p}}{\partial n} = \frac{2\rho}{m}\bar{p} \quad .$$

The present equation will be the kernel of this section. It appears to be useful to rewrite this equation in the equivalent shape

$$\frac{\partial \ln \bar{p}}{\partial n} = \mu s\left(\frac{1}{s+i\omega_0}+\frac{1}{s-i\omega_0}\right) \quad , \tag{6}$$

in which the constant m is defined by

$$\mu = \frac{\rho}{m} \quad . \tag{7}$$

It is useful to make the substitution

$$w = is \quad , \tag{8}$$

which maps the complex s-plane on the complex w-plane by a rotation of the s-plane over a quarter of a turn to the left. The frequency axis of the s-plane is mapped on the real axis in the w-plane. Positive values at this axis correspond to points at the negative frequency axis in the s-plane. The negative part of the real w-axis is the image of the positive frequency axis in the s-plane. As a result of this substitution (6) takes a shape which is rather convenient to work with. The equation has the shape

$$\frac{\partial \ln \bar{p}}{\partial n} = \mu w\left(\frac{1}{\omega_0+w}-\frac{1}{\omega_0-w}\right) \quad . \tag{9}$$

Now we are in a position to specify the place of the basilar membrane in the plane and to define ω_0 along the membrane. We assume that the basilar membrane falls together with the negative real axis of a complex z-plane. Its high frequency part starts at minus infinity and ends at the origin. The resonance frequency varies linearly according to $-Qx$. The positive constant Q can be met within time scaling. Therefore it is sufficient to put $Q=1$, so that

$$\omega_0(x) = -x \quad , \quad -\infty < x \leq 0 \quad . \tag{10}$$

We first restrict ourselves to positive real values of w. Insertion of (10) in (9) yields

$$\frac{\partial \ln \bar{p}}{\partial n} = \mu\left(\frac{1}{\frac{x}{w}+1}-\frac{1}{\frac{x}{w}-1}\right) \quad ; \quad -\infty < x \leq 0 \quad , \quad y = 0 \quad . \tag{11}$$

The right member of (11) possesses singular points at the real axis of the z-plane. The place of these points follows from

$$\frac{x}{w} = \mp 1 \quad .$$

For our problem the singularity with the minus sign is of special importance because it represents the point of resonance at the membrane. At the left side of this point the stiffness dominates. Between resonance and the origin the stiffness is of secondary importance and there we could put a zero condition for the pressure. The problem which results from this assumption will be solved at the end of this section. Here, we will first look for some basic properties which follow from (11).

In (11) the abscissa which determines resonance comprises the weight $1/w$. This factor determines the actual place of resonance at the membrane as a function of the frequency. We shall assume that the unknown ordinate includes the same weight. In that case it is natural to replace the normal n by y/w. Therefore we put

$$\frac{\partial}{\partial n}=\frac{\partial}{\partial\left(\frac{y}{w}\right)} \quad .$$

In consequence of this, the pressure $\overline{p}$ in (11) can be considered as a function at the real axis of a scaled z plane. We shall denote this plane as the $\zeta=\xi+i\eta$ plane, so that

$$\zeta=\frac{z}{w} \quad . \tag{12}$$

Next we conceive $\overline{p}$ as the limiting case of a function of z, so that at real axis $\overline{p}(\xi,w)=\overline{p}(\xi+i0,w)$. Because at this axis

$$\frac{\partial\overline{p}}{\partial\eta}=i\frac{d\overline{p}}{d\varsigma} \quad ; \quad -\infty<\xi<\infty \ , \quad \eta=0 \ ,$$

equation (11) can be integrated immediately. The result reads

$$i\ln\overline{p}(\zeta)=\mu\ln\frac{\zeta+1}{\zeta-1} \quad ; \quad -\infty<\xi<\infty \ , \quad \eta=0 \ , \tag{13}$$

where we omitted for the sake of convenience the constant of integration. It is well known from classical applications (for instance Spiegel, 1964) that the right member of (13) represents an orthogonal co-ordinate system in the complex z-plane. This system determines the amplitude and phase of the pressure at the real axis of the z-plane. In order to find these quantities it is customary to express the right member of (13) in terms of local polar co-ordinates with respect to the points $\varsigma=\mp 1$. Here we confine ourselves to write in (13) the logarithm as

$$\ln\left(\frac{\zeta+1}{\zeta-1}\right)=\ln\left|\frac{\zeta+1}{\zeta-1}\right|+i\arg\left(\frac{\zeta+1}{\zeta-1}\right) \quad .$$

Then, curves which determine amplitudes for the pressure belong to the family

$$\arg\left(\frac{\zeta+1}{\zeta-1}\right)=\alpha \ , \tag{14}$$

in which α is a constant which varies from curve to curve. It appears that members of this family are the circles

$$\xi^2+(\eta+\cot\alpha)^2=\frac{1}{\sin^2\alpha} \ .$$

Curves which determine the phase constitute the family

$$\ln\left(\frac{\zeta+1}{\zeta-1}\right)=\beta \ , \tag{15}$$

in which β is a constant which again varies from member to member of the family. This phase family consist of the circles

$$(\xi-\coth\beta)^2+\eta^2=\frac{1}{\sinh^2\beta} \ .$$

Both families together constitute the well known meshwork of Apollonius which has been given in Figure 1. In this figure the heavy lines are the iso-amplitude curves. Dashed curves are curves of equal phase. In the upper half-plane the range of α is $0\le\alpha\le\pi$.

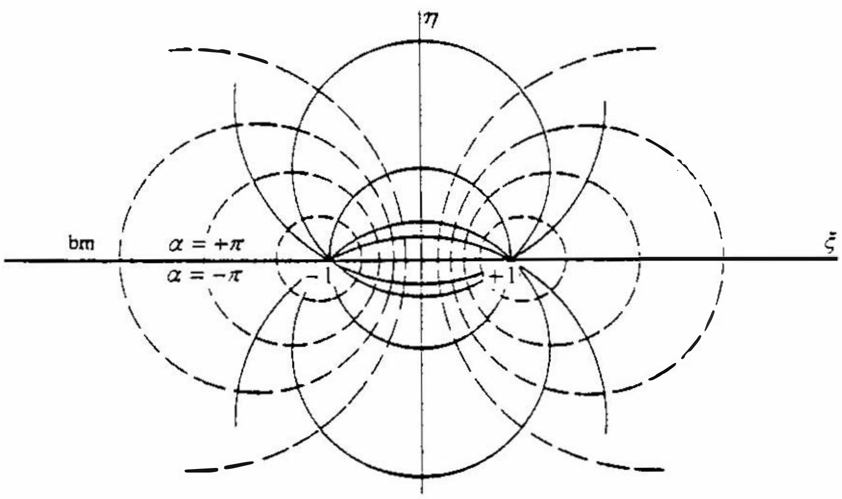


Figure 1. The classical co-ordinate system which is build up from circles of Apollonius and their orthogonal trajectories. This system determines the amplitude and phase of the pressure according to (13). The same system can be used as an intrinsic co-ordinate system in a boundary value problem which leads to an uneven behaviour of the pressure with respect to the membrane axis.

At the unit circle we have $\alpha = \pi / 2$. For $\alpha = \pi$ and $\alpha = 0$ the corresponding circles have been degenerated and fall together with the real axis. In the under half-plane the range of α is $-\pi \leq \alpha \leq 0$. In this case too, both equal signs correspond to circles which have been degenerated and fall together with the real axis. The value $\alpha = -\pi / 2$ corresponds to the unit circle. The range of the phase β varies according $-\infty < \beta < +\infty$. In the left half-plane β is negative, whereas in the right half-plane its sign is positive. The imaginary axis represents the degenerated circle $\beta = 0$. The points -1 and $+1$ are the degenerated circles $\beta = -\infty$ and $\beta = +\infty$, respectively.

According to (13), (14) and (15) the pressure can be written as

$$\bar{p}(\alpha, \beta) = \exp(\mu(\alpha - i\beta)) \ ,$$

where α and β follow from Fig. 1. The negative real axis represents the basilar membrane. That part of the membrane axis which extends from minus infinity up to and included the point of resonance, is a cut in the complex ζ-plane. Across this cut the pressure is discontinuous. At the upper-side of this axis and near the point of resonance the amplitude of the pressure shows discontinuous behaviour. At the left side of resonance the amplitude equals $\exp(\mu\pi)$ and diminishes suddenly to 1 at the other side of resonance. This implies that the difference between the amplitude levels before and after resonance equals $20\mu\pi \log e$ dB. The constant μ is defined in (7). A typical value of this quantity is 20. Thus the difference between the levels exceeds 500 dB. Ranke (1942) was the first who noticed this discontinuous behaviour. After him, a considerable better description of this phenomenon has been given by de Boer (1979, 1984).

The quantity $-\beta$ is the phase of the pressure at the membrane. β tends to minus infinity as the distance to the point of resonance tends to zero. The slope of $-\beta$ is positive at the left side of resonance and negative at the other side. In consequence of this, the pressure at both sides of resonance represents a wave which travels always to the point of resonance. Moreover, de Boer showed from this near resonance behaviour that the point of resonance actually models a 'sink' for the corresponding energy in the direct environment of the point of resonance.

From the difference between the levels before and after resonance follows that the pressure is effectively zero between the point of resonance and the origin of the plane. When the pressure in that region should be zero indeed, we expect that the pressure at least in that region is (almost) uneven with respect to the membrane axis. The present solution for the pressure fails to describe this property even approximately. Direct inspection shows that the pressure is inverted when values of α and β at the lower side of the membrane axis are used.

In order to solve this imperfection, we define a new boundary value problem in which α and β are considered as functions which determine a curvilinear co-ordinate system in the ς-plane. At the left side of resonance we prescribe at the membrane the present shape of the solution for the pressure. Between resonance and the origin of the plane we put the new boundary condition $p = 0$ at the membrane axis.

Next, we reflect these boundary conditions with respect to the imaginary axis of the ς-plane. Then we arrive at a boundary value problem which in terms of α and β reads

$$
\begin{aligned}
&\Delta\bar{p} = 0 && 0<\alpha<\pi\ ,\quad -\infty<\beta<+\infty\ ; \\
&\bar{p} = e^{\mu\pi}\left(\cos\mu\beta - i\sin\mu\beta\right) && \alpha=\pi\ ,\quad -\infty<\beta<+\infty\ ; \\
&\bar{p} = 0 && \alpha=0\ ,\quad -\infty<\beta<+\infty\ .
\end{aligned}
\tag{16}
$$

The solution for this problem is

$$\bar{p} = \frac{e^{\mu\pi}}{\sinh\mu\pi}\sinh\mu\alpha\left(\cos\mu\beta - i\sin\mu\beta\right)\ . \tag{17}$$

It is easy to verify that this solution complies with the terms of problem (16). Let us extend the present solution over the whole α,β-plane. Then, because α is uneven with respect of the line of symmetry $\alpha=0$, it is readily seen that (17) has a second important property

$$\bar{p}(\alpha,\beta) = -\bar{p}(-\alpha,\beta)\ , \tag{18}$$

which holds true even to the left-hand side of the point of resonance. Therefore, as a result of this kind of modelling, the pressure difference across the membrane in the region where the stiffness dominates is twice the pressure at the upper-side of the membrane too. This uneven behaviour, which includes the membrane as a discontinuity as far as resonance, leads to the possibility to apply the principle of reflection to construct a half infinite strip-like model so that the pressure vanishes at boundaries at a distance h from the basilar membrane. At this stage we will not carry out this process. For, the purpose of this work is primarily to model basilar membrane properties and not to construct complete cochlear models.

Essentially, the present solution for the pressure follows from (11). We studied this equation for positive real values of w. When w is negative - i.e. $w=-|w|$ - we expect that the pressure $\bar{p}$ which follows from this equation is the complex conjugate counterpart of (13). This requirement has only been satisfied when the pressure $\bar{p}$ at the negative real axis is conceived as a function of the complex conjugate co-ordinate $\bar{\zeta}$. In consequence of this we have to conceive the pressure $\bar{p}$ at the membrane for negative values of w as $\bar{p}(\xi,w)=\bar{p}(\xi-i0,w)$. Then, when the same way of reasoning of the first part of this section is followed, the ultimate result reads

$$-i\ln\bar{p}\left(\bar{\zeta}\right) = \mu\ln\frac{\bar{\zeta}+1}{\bar{\zeta}-1}\quad,\ \text{with}\ \bar{\zeta} = \frac{Q}{|w|}\bar{z}\ . \tag{19}$$

It can be shown that the pressure which follows from (19) is indeed complex conjugate to the pressure according to (13). We shall not work out a problem similar to (16) for the conjugate pressure. This is not necessary. For, when in (17) i is replaced by $-i$ the complex conjugate counterpart is found immediately and possesses the required symmetry properties.

3. Properties at a fixed point

In section 2 we studied the pressure at the membrane as a function of the length parameter along the membrane for a fixed value of the frequency. In this section we will change the role of the fixed and varying quantity. The main contribution to the pressure at the membrane is determined by (13). We will make a start from this expression. Let us insert (12) in this formula. Then it appears that (13) takes the shape

$$i \ln \bar{p}(w) = \mu \ln\left(\frac{x+w}{x-w}\right) . \tag{20}$$

The point x is the fixed place. Until now we only considered this formula for positive values of w. It can be shown that for negative values of w the pressure $\bar{p}(w)$ according to (20) is complex conjugate to its values for positive values of w and corresponds to (19) from the previous section. Therefore, in this section it is not necessary to distinguish between different expressions for positive and negative values of w. At the membrane the resonance frequency equals $\omega_0(x) = -x$, $-\infty < x \leq 0$ and according to (8) $w = is$. Then it is readily found that an equivalent shape for (20) at a fixed point at the membrane reads

$$\ln \bar{p}(s) = i\mu \ln\left(\frac{s - i\omega_0(x)}{s + i\omega_0(x)}\right) . \tag{21}$$

Next we scale the complex s - plane with $\omega_0(x)$. In consequence of this, (21) takes the elementary shape

$$\ln \bar{p}(s) = i\mu \ln\left(\frac{s-i}{s+i}\right) . \tag{22}$$

In order to investigate this expression at the frequency axis of the s - plane we put as usual

$$s - i = r_1 \exp(i\varphi_1) \quad \text{and} \quad s + i = r_2 \exp(i\varphi_2) ,$$

so that (23) reduces to

$$\ln \bar{p}(s) = \mu(\varphi_2 - \varphi_1) + i\mu \ln \frac{r_1}{r_2} \ . \tag{23}$$

At the ω - axis of the s - plane, the first term of the right-hand side of (23) equals

$$\ln|\bar{p}(\omega)| = \begin{cases} \mu\pi & \text{for} \quad |\omega| < 1 \\ \\ 0 & \text{for} \quad |\omega| > 1 \ . \end{cases} \tag{24}$$

This expression determines the amplitude of the pressure. The imaginary part of (23) at the ω - axis can be written as

$$\arg(\bar{p}(\omega)) = \mu \ln\left|\frac{\omega - 1}{\omega + 1}\right| \ . \tag{25}$$

This argument determines the phase behaviour. The last two expressions are the normalised characteristics for the amplitude and the phase of the pressure respectively, according to (21) at an arbitrary point of the membrane. The next figure shows the plots of both characteristics.

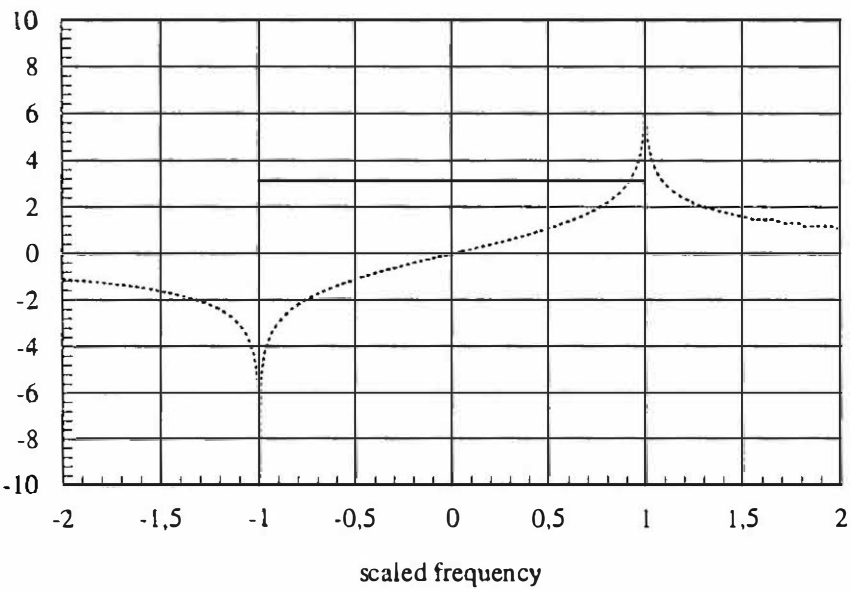


Figure 2. Normalised amplitude and phase according to (22) for the pressure at an arbitrary point of the membrane. The solid line is the amplitude characteristic. This line follows from (24). The dotted line is the phase as follows from (25). The scaling factor is the resonance frequency of the point under consideration. In order to suppress the mathematical singular behaviour, the place of the singularities in the s - plane has been shifted over a small distance to the left.

The singular behaviour of (22) at $s = \pm i$ is the cause of the discontinuity for the amplitude of the pressure and the weak singular behaviour of the phase. This extreme behaviour has never been observed in measurements. Therefore, in the next section we shall suppress these 'undesired' effects in a rather pragmatic way. At this stage we again note that near resonance the 'drop-shot' of the amplitude is rather large. In correspondence to the behaviour of the pressure in the place domain, the jump of the pressure amplitudes corresponds to a factor $\exp(\mu\pi)$ and equals a difference of $20\mu\pi \log e$ dB between the levels just before and after resonance. In a numerical sense this means that after resonance the pressure is negligible. Therefore, in practice it is allowed to restrict ourselves to frequencies $|\omega| < 1$.

From (24) and (25) the normalised impulse response can be calculated. We applied straightforwardly the inversion formula for Fourier transforms. Here it is sufficient to restrict ourselves to the result. This is shown in the next figure.

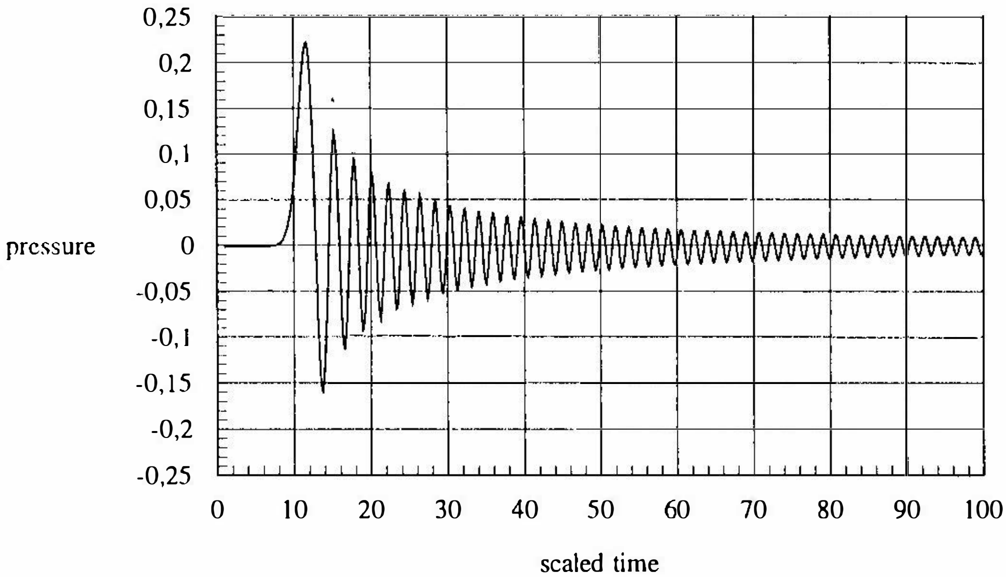


Figure 3. Impulse response of the filter described by the characteristics (24) and (25). In this figure the time axis has been scaled with the resonance frequency at the point of consideration.

Inspection of this response shows that it is as if we deal with a causal signal which has been delayed over a well defined time τ. However, according to the 'time-shift' rule from Laplace transforms (Spiegel, 1965) the expected shape of (22) would be

$$\bar{p}(s) = \exp(-\tau s)q(s)$$

where $q(s)$ tends to zero as s tends to infinity. Apparently this is not the case. For, it is easy to verify from (22) that when s tends to infinity, in the limiting case $\bar{p}(s)$ equals 1. This limit for $\bar{p}(s)$ points to the presence of a delta function at the origin of the time scale. Because $\bar{p}(s)$ is regular in the right hand side of the s-plane but does not tend to zero for large values of s, we therefore conclude that the impulse response starts with a low level delta function at the time $t = 0$.

We determined the delay time of the filter. Quite formally, the delay time τ of a filter is defined as (see for instance: Papoulis, 1987) minus the rate of change of the phase at $\omega = 0$. Thus

$$\tau = -\lim_{\omega \to 0} \frac{d \arg(\overline{p}(\omega))}{d\omega} \quad .$$

Then follows from (25) that $\tau = 2\mu$. This formal expression for the delay time resembles the delay of the front of the impulse response of the system.

Next we shall show that this delay time can be considered as a pseudo front delay and is approximately present in the transform (22). In order to do that, we first note that for $|s/i| \leq 1$ and $|s/i| \neq \pm 1$ the following series expansion holds true

$$\frac{1}{2} \ln \frac{s-i}{s+i} = i\left(\frac{\pi}{2} + s \div \frac{s^3}{3} + \ldots\ldots\right) \quad . \tag{26}$$

Then follows from (22) and (26) that $\overline{p}(s)$ within the region of convergence of this expansion can be written as

$$\overline{p}(s) = e^{-2\mu s} \overline{q}(s) \quad ,$$

in which

$$\overline{q}(s) = \exp\left(2\mu\left(-\frac{\pi}{2} + \frac{s^2}{3} - \ldots\ldots\right)\right) \quad .$$

This shape is restricted to the range over which the series expansion (26) can be justified. However, this range determines the main contribution of the spectrum at the frequency axis. Therefore, in an approximate sense the quantity $\tau = 2\mu$ can be considered as the delay of a pseudo front. After re-scaling of the s - plane, the pseudo front delay time takes the shape

$$\tau(x) = \frac{2\mu}{\omega_0(x)} \quad . \tag{27}$$

The constant μ is defined by (7). When μ is inserted in (27) we arrive at an equivalent reading for the pseudo front delay time at the point x of the membrane in the shape

$$\tau(x) = 2\frac{\rho}{\sqrt{m\kappa(x)}} \quad .$$

The next figure is from Van Dijk (1990) and shows $\tau(x)$ according to (27) with $\mu = 20$ and cochlear nerve delay times for the chinchilla after Ruggero and Rich (1987).

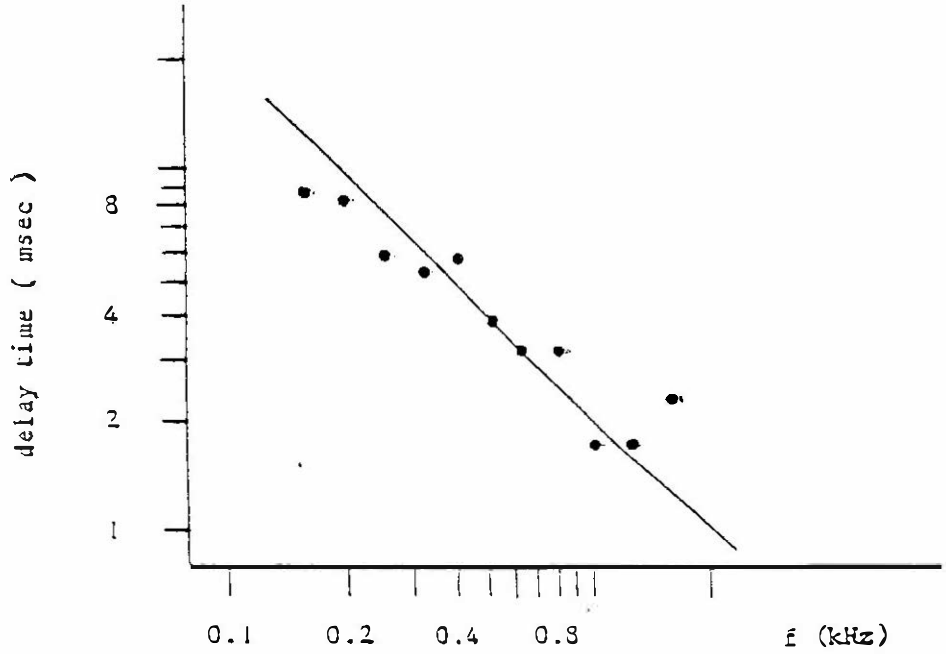


Figure 4. Cochlear nerve delay times as a function of the frequency according to measurements from Ruggero and Rich (1987). Dots are results of measurements. The straight line represents the pseudo front delay according to (27) with $\mu = 20$.

In the next section we will develop approximate filters which are applicable in practice and constitute a filter bank for the basilar membrane as a pre-processor.

4. Approximate filters

In the previous section we pointed to the unnatural behaviour of the pressure near resonance. In Fig. 2 the frequency characteristics at resonance show the extremely abrupt behaviour for the amplitude and a phase function which is singular. Both effects can be suppressed when a small amount of damping is introduced in the system. This is accomplished when the singular points $s = \pm i$ are shifted over a small distance to the left in the s - plane. However, this only results in typical local effects such as a small rounding at the edges of the amplitude characteristic and local suppression of the singular behaviour of the phase. Despite these positive effects all global characteristics remain conserved and remain therefore still unnatural. Another way to control these effects is to replace the singular functions with functions which approximate this singular behaviour. This can be done in the following way. In the s - plane the relative change of rate of the pressure $\overline{p}(s)$ is found after differentiation of (24). The result is

$$\frac{d \ln \overline{p}(s)}{ds} = i\mu\left(\frac{1}{s-i} - \frac{1}{s+i}\right). \tag{28}$$

The points $s = \pm i$ are first order poles of the right member of (28). These poles essentially are the cause of the discontinuity of the amplitude and the singular behaviour of the phase at resonance. In order to suppress these effects, we propose to replace both terms between the brackets in (28) by functions which approximately simulate similar singular behaviour.

Let us assume for a moment that the origin of the s-plane is a first order pole in the s-plane. Then we can profit from the approximation that for sufficiently small values of c holds

$$\frac{1}{s} \approx \frac{1}{2ic} \ln \frac{s+ic}{s-ic} \quad . \tag{29}$$

The meaning of (29) is that the pole at the origin can be conceived as the derivative of a dipole which has been located at the frequency axis. The distance between the internal poles of the system is $2c$. The idea of (29) can be applied to both terms between the brackets in (28). This yields the approximation

$$\frac{d \ln \bar{p}(s)}{ds} = \frac{\mu}{2c} \left(\ln \frac{s-s_2}{s-s_1} - \ln \frac{s-s_4}{s-s_3} \right) ,$$

in which

$$\begin{aligned} s_1 &= \mathrm{i}(1+c) \quad & s_2 &= \mathrm{i}(1-c) \\ s_3 &= -i(1-c) \quad & s_4 &= -i(1+c) \ . \end{aligned} \tag{30}$$

Then, integration over s leads straightforwardly to

$$\begin{aligned} \ln \bar{p}(s) = -\frac{\mu}{2c} \big((s-s_1)\ln(s-s_1) - (s-s_2)\ln(s-s_2) \\ - (s-s_3)\ln(s-s_3) + (s-s_4)\ln(s-s_4) \big) \ , \end{aligned} \tag{31}$$

For the sake of convenience the constant of integration has been neglected. Inspection of (31) shows that indeed the singular behaviour has been suppressed. For, essentially (31) has been composed of terms of the kind $s \ln s$ and it is known from analysis that when s tends to the origin every positive power of s tends faster to zero than $\ln s$ tends to infinity near that point.

In order to develop the amplitude and phase characteristics at the frequency axis, we express as usual the terms $s - s_j$ in local polar co-ordinates. Thus

$$s - s_j = r_j \exp(i\varphi_j) \ , \ j = 1, \ldots, 4 \ .$$

Then, as follows after some calculations, the amplitude frequency characteristic for the pressure obeys the expression

$$\ln\left|\bar{p}(\omega)\right| = \frac{\mu}{2c}\Big((\omega-\omega_1)\varphi_1 - (\omega-\omega_2)\varphi_2 \\ -(\omega-\omega_3)\varphi_3 + (\omega-\omega_4)\varphi_4\Big) \ , \tag{32}$$

where

$$\varphi_j = \begin{cases} \dfrac{\pi}{2} & \text{for} \quad \omega > \omega_j \\ -\dfrac{\pi}{2} & \text{for} \quad \omega < \omega_j \end{cases} \qquad j = 1, \ldots\ldots, 4$$

and

$$\omega_1 = 1 + c \qquad \omega_2 = 1 - c$$
$$\omega_3 = -1 + c \qquad \omega_4 = -1 - c \ .$$

The phase characteristic which follows from (31) reads

$$\arg\left(\bar{p}(\omega)\right) = -(\omega-\omega_1)\ln\left|\omega-\omega_1\right| + (\omega-\omega_2)\ln\left|\omega-\omega_2\right| \\ +(\omega-\omega_3)\ln\left|\omega-\omega_3\right| - (\omega-\omega_4)\ln\left|\omega-\omega_4\right| \ . \tag{33}$$

The next figure is a plot of (32) and (33).

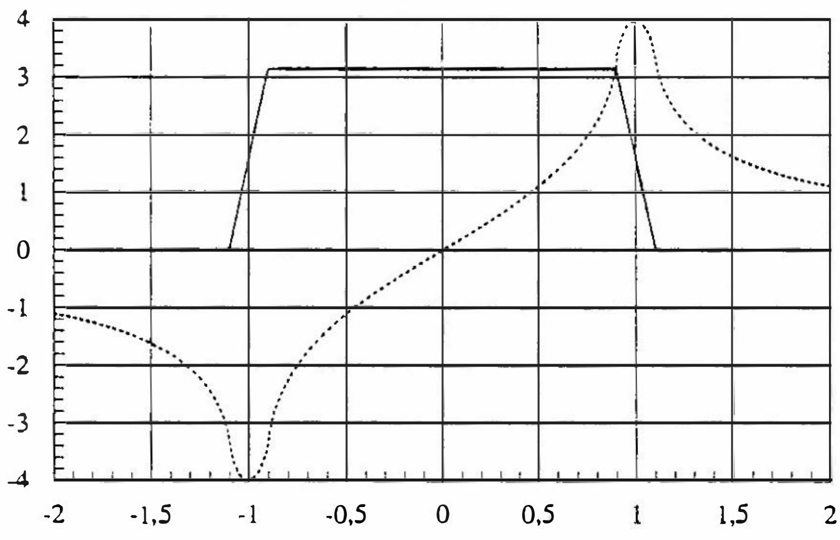


Figure 5. Normalised amplitude and phase characteristics for the pressure at points of the basilar membrane. The solid line is the amplitude characteristic as follows from (32). The dotted line is the phase according to (33). The well defined slope near resonance is the result of an approximation in which the first order poles in the relative change of the pressure (28) have been replaced by dipoles. The dipole distance is a 'free' parameter and controls the 'high frequency' slope near resonance.

Let us compare Fig. 2 and Fig. 5 with each other. It appears that the jump at resonance has been replaced by a linear decreasing function between the poles of the dipole. The steepness of the slope depends on the distance $2c$ between the poles of the dipole. If this distance is sufficiently small, the amplitude behaviour resembles the original discontinuity. We shall conceive the dipole distance $2c$ as a 'free' parameter which can be used to simulate a certain amount of damping.

In the dipole construction singular behaviour is suppressed. In consequence of this, the phase according to (33) is continuous at every point of the frequency axis. The range over which the phase changes again depends on the dipole distance $2c$. The next figure shows the response according to the dipole approximation of the pressure for a moderate value of $2c$.

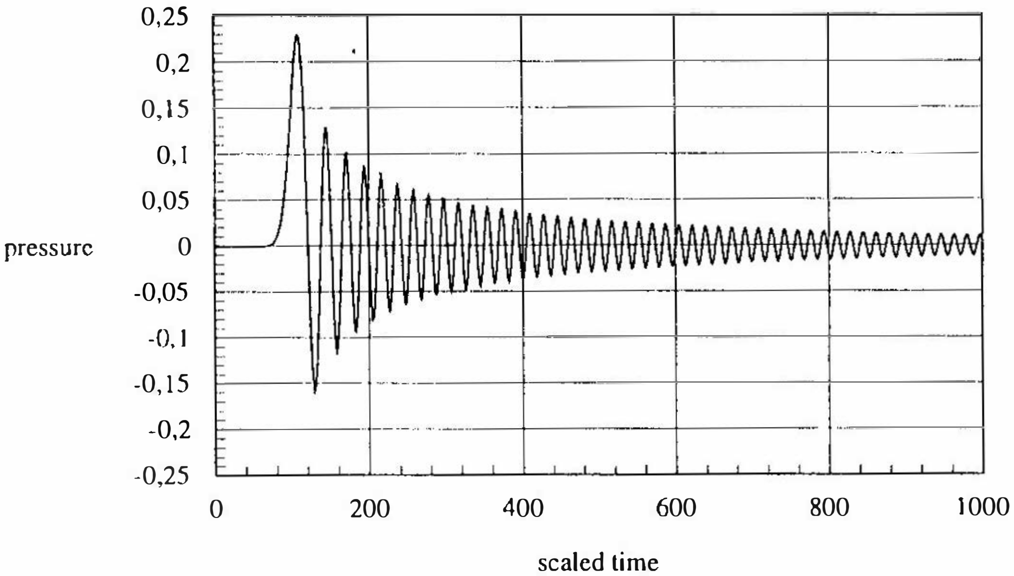
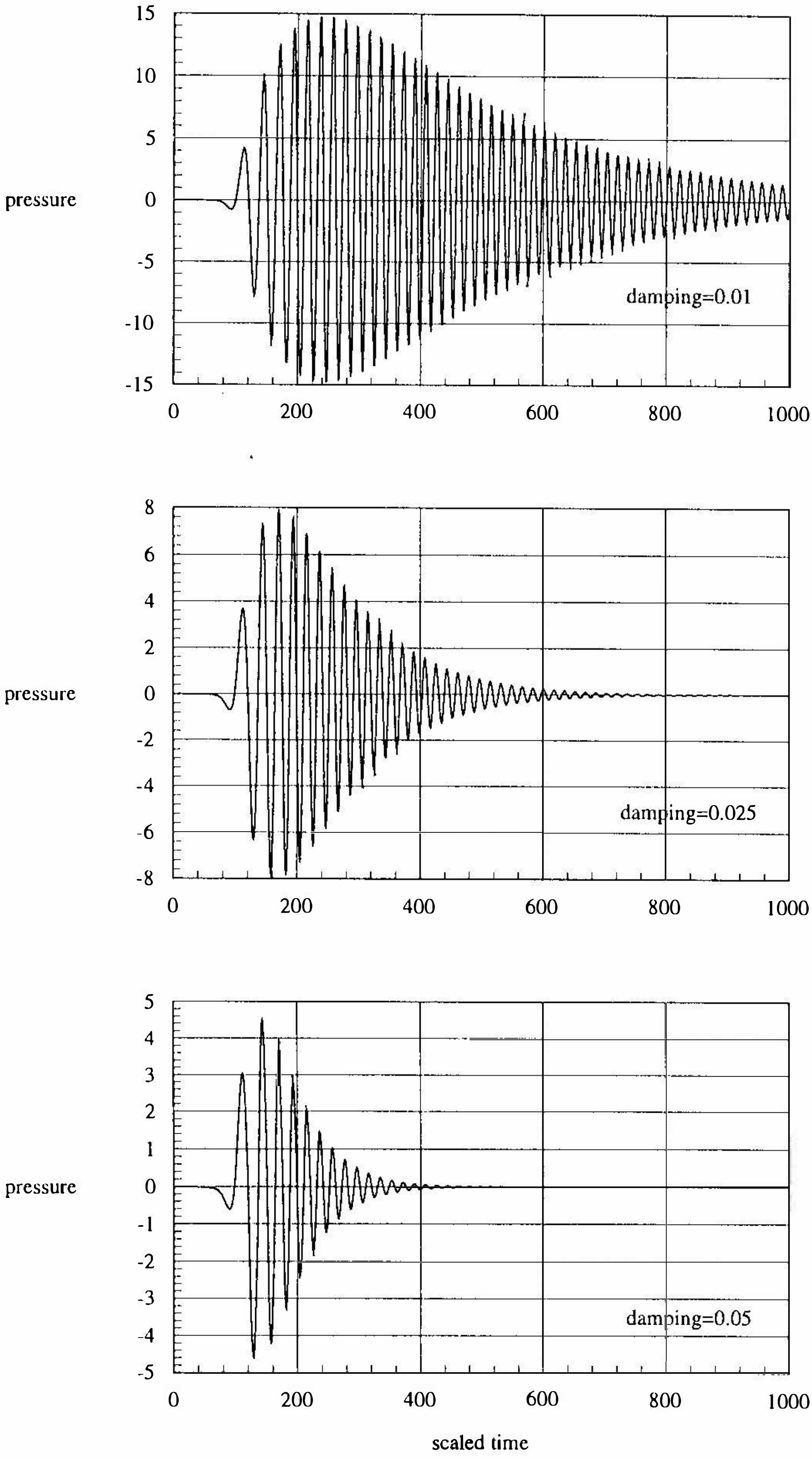


Figure 6. Impulse response of the filter described by the characteristics (32) and (33) and Fig. 5 at a small value for the damping. The pseudo front delay time according to (27) remains clearly visible.

The front delay time which is observable in the response corresponds to (27). Just after this front, the time difference between the first two peaks is relatively large. The time difference between successive peaks diminishes as the time proceeds. Thus, this signal models a chirp-like sound. This response is the stimulus for the local filter at a point of the membrane. Because this last filter responds according to the well known characteristics of a second order mass spring system, the ultimate response resembles the response in Fig. 6 closely. Therefore, at this stage we shall not yet pay attention to this last response. In Fig. 6 the magnitude of the large peaks just after the front is mainly determined by the low frequency components in the spectrum of the pressure (Fig. 5). When the slope in the amplitude spectrum before resonance is varied from the present zero value to positive values, the high peaks near the front are suppressed and the response tends to resemble actually measured low level characteristics for the velocity according to Ruggero et al. (1992). The next figure shows these responses according to the present approximation and results of measurement.


15
10
5
pressure
0
-5
-10
-15
damping=0.01
0
200
400
600
800
1000
8
6
4
2
pressure
0
-2
-4
-6
-8
damping=0.025
0
200
400
600
800
1000
5
4
3
2
1
pressure
0
-1
-2
-3
-4
-5
damping=0.05
0
200
400
600
800
1000
scaled time

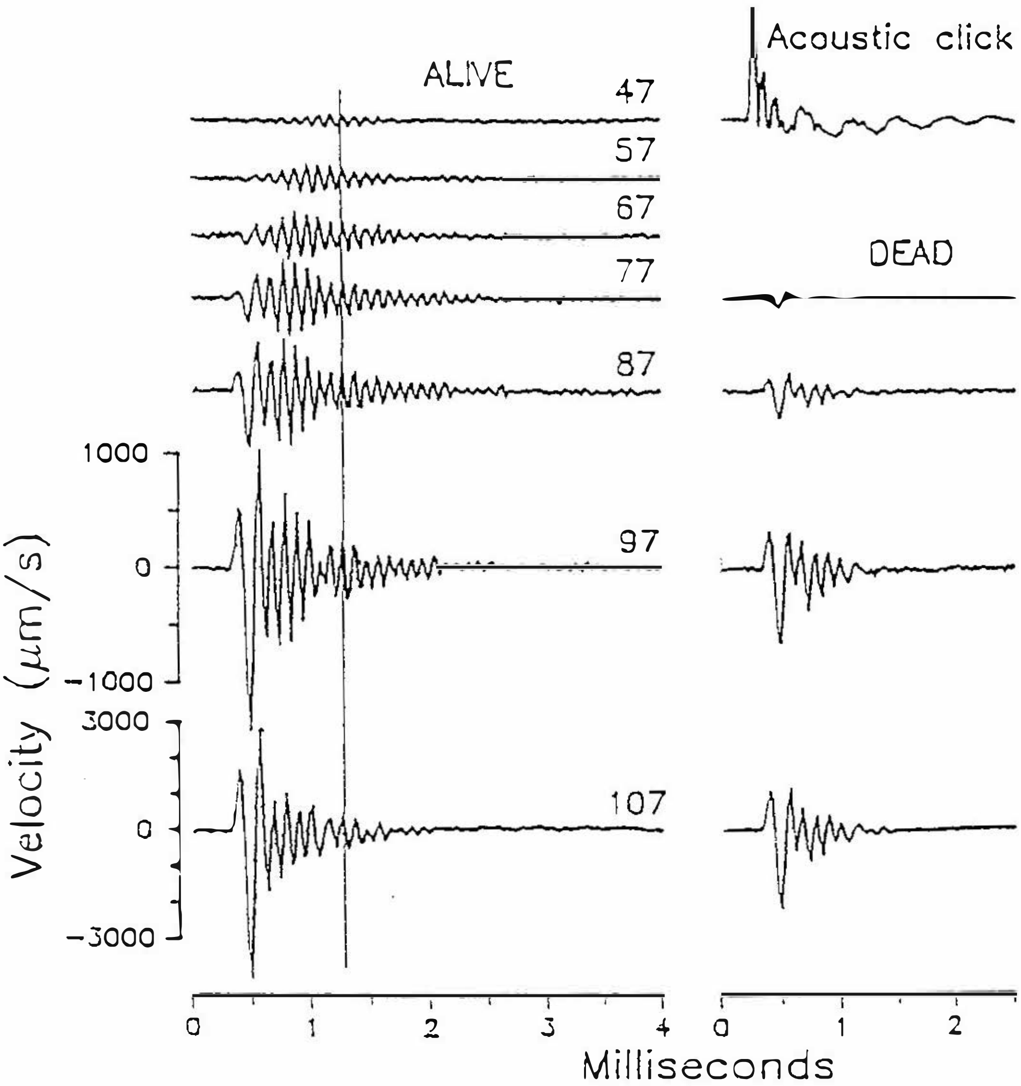


Figure 7. Impulse responses of the velocity for three different values of the damping according to the present approximation and results of measurements from Ruggero at al. (1992).

There are several objections against the direct applicability of the present characteristics.

First of all, the analytical approach of the present and previous sections mainly models near resonance behaviour. This follows directly from equation (11). Therefore, the question concerning the influence of different stiffness functions on the responses is still an open one.

Secondly, in order to 'force' a solution of equation (9) we even scaled the normal in this equation with respect to the frequency so that an analytical solution could be found. This can have an influence on quantitative properties of the characteristics which are not negligible.

Thirdly, until now we neglected the constant of integration in the solution of the first order equation (9) for the famous reason of 'convenience'. However, this constant can be used to relate the behaviour of a point of the basilar membrane to a prescribed pressure at the stapes.

5. Discussion and conclusions

In section 2 we studied some properties of the membrane condition in the lossy case. This condition can be written in the shape (6) or (9). Then it is clear that the normal derivative of the logarithm of the pressure has two poles. These poles are the singularities of the problem and determine the points of resonance in the place domain.

In mathematics it is common knowledge that the behaviour of a solution for a problem in which singularities are present follows from the characteristics of the problem near those points. This is an indication to restrict ourselves to only those singular points which are close to the basilar membrane. In consequence of this, we paid attention to a simplified version of the problem, so that the properties which follow from the point of resonance near the membrane can be easily found. This approximation corresponds to the notion that these points are the physically relevant points of resonance of the problem.

The place of a point of resonance near the membrane depends on both the frequency and the damping. This last parameter is responsible for the distance of the mathematical point of resonance to the basilar membrane. However, the sign of the frequency under consideration determines whether the point of resonance is found at the upper side of the membrane or at the lower side. Therefore, in order to study near resonance effects adequately in the place domain, we have to distinguish between positive and negative frequencies and in consequence of this to an upper-plane or a lower-plane approximation to the problem. In the lossless case the points of resonance coincide at the membrane axis. However, because the lossless case is the limiting case of the lossy one, the distinction between an upper- and a lower-plane approximation must remain conserved. The upper-plane and the lower-plane approach to the membrane axis yield expressions for the complex amplitudes for harmonic vibrations proportional to $\exp(+i\omega t)$ and $\exp(-i\omega t)$. The respective amplitudes are indeed complex conjugate to each other. This is an indication that the proposed distinction is the only correct way to reach this goal.

In the main part of this section we considered the membrane condition for the pressure as an equation in the lossless case. Nevertheless, we reckoned systematically with the necessary distinction between positive and negative frequencies. As far as we know no other authors paid attention to this point.

It appeared that along the membrane the distance from the stapes to the point of resonance is mainly determined by the scaled length parameter. Here, the scaling factor is the frequency under consideration. When the normal to the membrane near the point of resonance is chosen in agreement with this behaviour, it follows that the membrane condition can be integrated straightforwardly.

In the last decades, only Siebert (1974) and De Boer (1979; 1984) paid attention to properties of the basilar membrane condition which are related to the present approach. Siebert (1974) started to study this problem, partially in a numerical way. De Boer (1979) considered the lossless case. After the application of some analytical means, he argued that in the direct vicinity of the point of resonance the solution for means, he argued that in the direct vicinity of the point of resonance the solution for the pressure represents a wave which must travel towards the point of resonance. This property is present in all solutions of this chapter too.

There is no clear evidence whether the pressure after resonance must represent a wave which travels towards the point of resonance or not. In general, the levels of the membrane motion after the point of resonance are extremely small. In consequence of this, it is almost impossible to make hard decisions on this question. Nevertheless, after the present current opinion, points of the basilar membrane perform a motion in almost the same phase. In that case it is as if the solutions of this work bear the intrinsic imperfection of a travelling wave after resonance. There are arguments to suppress or to remove this property from the present solutions. For, at the derivation of the membrane condition we assumed that the pressure difference across the membrane consists of two terms which are an uneven reflection of each other. This assumption reduces the difference to only one term which is twice the pressure at the upper-side of the membrane. It is unlikely that this property is a true natural characteristic. According to our opinion near symmetrical physical laws govern nature rather than pure symmetrical ones. However, symmetry often points to an essential feature of the law and frequently leads to a description of main characteristics of a problem under consideration. When in our problem the pressure difference across the membrane consists of two terms characterised by a broken symmetry, there must be an additional term which has been neglected until now. This term must originate from the difference between two terms which are not exactly anti-symmetrical. However, because in our opinion the anti-symmetrical parts lead to the main contribution of the pressure difference, the magnitude of the remaining term cannot be overwhelming. As a rough local guess which might be true near resonance we shall assume that the difference must be extended with a small constant. This is sufficient to suppress the travelling wave character of the solution after resonance at the favour of a membrane motion with the same phase.

The validity of these considerations can be justified, since it is allowed to add an arbitrary constant to the pressure in our problem. At this place it is sufficient to note that when p is the pressure which obeys the membrane condition, the same holds true for the pressure in the shape $p + const$. Because the additional constant must be very small, it is not very interesting to look for numerical consequences.

The solution of the membrane condition shows that after the point of resonance the level of the pressure is extremely small. This behaviour can be approximated effectively when we take the pressure between the point of resonance and the helicotrema as a vanishing one. In that case it is impossible to express the pressure both at the membrane and in the plane in terms of analytical functions. This is a necessary consequence of the principle of analytical continuation (see for instance Spiegel, 1965).
In section 2 we solved this problem in terms of 'almost' analytical functions. As an important additional result, it appeared that the non-analytical solution is uneven with respect to the y-direction of the problem. This property is sufficient to construct strip- or box-like models for the cochlea with two scalae with hard walls. Then, as a result of the way in which those models can be constructed, the pressure is an uneven

function with respect to the membrane. We will not carry out the construction of those models.
The main reason for this is that in this kind of extensions, the shape of the pressure in this chapter is always the main contribution to the pressure difference across the membrane.

The present approach is appropriate to describe scaled amplitude and phase characteristics for an arbitrarily point of the membrane. We first derived an explicit expression for the pressure at the membrane. In this expression the complex frequency s has been scaled with respect to the resonance frequency $\omega_0(x)$ at the point x of the membrane. In the resulting expression the singular points are found at the frequency axis of the complex s-plane. In the lossy case these points have been shifted over a small distance to the left. In consequence of this, there are no singular points in the right half-plane. Then, it holds that the function under consideration belongs to the class of minimum phase functions. De Boer (1997) and De Boer and Nuttal (1996) argued that reponses from points of the basilar membrane can be matched by responses from minimum phase filter functions.

We determined the impulse response from the characteristics which directly follow from the analytical approach. It is as if the response at a point x has been delayed over a characteristic time $\tau(x)$. This time is proportional to the density of the fluid and inversely proportional to the square of the mass of the membrane and the stiffness. When typical parameters are used, this time corresponds to the cochlear nerve delay time as has been measured by Ruggero (1987).

A better analysis shows that the delay time which follows from the model, can be conceived as the front delay time for the pressure wave at the membrane. At this place we point to the property that the time difference between a click at the eardrum and the response in the shape of a resulting echo as has been measured by several authors (for instance: Kemp, 1978; Wit and Ritsma, 1980). It appears that these measurements can be predicted very well by the present pseudo front delay time $\tau(x)$. This corresponds to the notion that once the travelling wave has reached the point under consideration, outer hair cells are stimulated. Then, as follows from the incompressibility of the fluid, effects as a result of this must be observable almost immediately at the entrance of the system.

The impulse response for the deflection or the velocity which has been calculated directly from the frequency characteristics has been composed of frequency components with frequencies up to the resonance frequency of the point. Inspection of this impulse shows that it actually represents a chirp-like 'sound'. When this response is compared with results of measurements (Ruggero, 1992), it appeared that the theoretical response resembles responses measured at high stimulus levels better than low level responses. From this last responses follows that in the theoretical spectrum of the impulse response, the amplitudes of low frequency components are too large compared with amplitudes of components near the resonance frequency. When in the theoretical spectrum amplitudes are weighted artificially, so that the near resonance components are favoured with respect to the low frequency ones, the resulting chirp-like response resembles measured responses. The present model does not give an explanation of this effect. In literature several proposals are found which suggest that there are additional forces - presumable caused by outer hair cell activity - which contribute to the well defined sharp low level response. De Boer (1983) proposed that the sharpening of this response is caused by an effective lowering of the damping, especially in a region near the point of resonance. Mammano and Nobili (1993) argued that the physical background of this lowering effect can be caused by outer hair cell activity which tends to diminish the influence of the internal viscosity of the organ of Corti. Both from a physical point of view and from reasons for modelling, the incorporation of outer hair cell activity in models of the cochlea is still an interesting unsolved question.

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